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ON (m,n)-ARY P-H_v-MODULES AND THEIR ISOMORPHISM THEOREMS

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Abstract. After introducing the definition of hypergroups by Marty, the study of hyperstructures and its generalization to (m,n)-ary hyperstructures has been of great importance. In this paper, we construct (m,n)-ary H_v-modules over (m,n)-ary H_v-rings by using P-hyperoperations. We study their properties and prove their isomorphism theorems.

1. Introduction

Hyperstructure theory was introduced for the first time in 1934 at the eighth Congress of Scandinavian Mathematicians, by Marty [22], as an extension of algebraic structures. Marty generalized the notion of groups by defining hypergroups. The class of algebraic hyperstructures is larger than that of algebraic structures where the operation on two elements in the latter is again an element whereas the hyperoperation of two elements in the first class is a non-void set. Since then, several articles and books were published on hyperstructure theory and its applications [1, 4, 8]. A special type of hyperoperations, known as P-hyperoperation was defined by Vougiouklis [32, 29, 30] and studied by several authors [27, 8, 25, 26, 34]. In 1991, Vougioklis [33] in the Fourth AHA Congress generalized hyperstuctures by introducing a larger class known as H_v-structures. A lot of work on some H_v-structures like H_v-groups, H_v-rings, H_v-modules, etc. was published [3, 2, 10, 15, 28].

On the other hand, the notion of n-group is another generalization of group. It seems that the first idea of investigations of n-ary algebras goes back to Kasner's lecture at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. But the first article concerning the theory of n-groups was written (under inspiration of Emmy Noether) by Dörnte [13] in 1928. In [11], Davvaz and Vougiouklis introduced the concept of n-hypergroups as a generalization of both hypergroups in Marty's sense and n-groups. Then this concept was studied by many authors, see, for

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example Ghadiri and Waphare [16], Leoreanu-Fotea and Davvaz [21, 19, 20], Davvaz et al. [6, 7, 9], Mirvakili and Davvaz [23].

New generalizations for algebraic structures were defined where the notion of n-ary algebraic structure was extended to the notion of n-ary H_v-structures. A link between P-hyperoperations and (m,n)-ary H_v-modules was established in [12], where Davvaz and Vougiouklis defined three kinds of external n-ary P-hyperoperations, and they used them to construct several (m,n)-ary H_v-modules. On the other hand, Al-Tahan and Davvaz [27] defined a new P-H_v-module over P-H_v -rings and studied several properties of it starting from the isomorphism theorem to the fundamental relation. In [12], Davvaz and Vougioklis defined (m,n)-ary P-H_v-modules over (m,n)-ary P-rings. In our paper, we generalize the definition in [12] by constructing a new (m,n)-ary P-H_v-module over (m,n)-ary P-H_v-ring, and study its properties. The remaining part is organized as follows: In Section 2, we present some definitions and results related to hyperstructures, (m,n)-ary hyperstructures and to P-hyperstructures. In Section 3, we construct (m,n)-ary H_v-modules over (m,n)-ary H_v-rings using P-hyperoperations, study their properties and present some examples. Finally, in Section 4, we prove the isomorphism theorems for (m,n)-ary P-H_v-modules.

2. Preliminaries

In this section, we gather all definitions we require for hyperstructures. We shall use the notation x_i^j to denote the sequence $x_i, x_{i+1}, \ldots, x_j$. Also, the sequence $\overbrace{a,\ldots,a}^{i}$ will be denoted by $a^{(i)}$. Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the family of all non-empty subsets of H. In general, for a positive integer n an *n-hyperoperation* on H is a mapping $f : H^n \to \mathcal{P}^*(H)$ where H^n denotes the set of n-tuples over H. If for all $(x_1,\ldots,x_n) \in H^n$, the set $f(x_1^n)$ is a singleton, then f is called an *n-operation*.

If $A_1,\ldots,A_n$ are non-empty subsets of H, then we denote

$$f(A_1,\ldots,A_n) = \bigcup\{f(x_1,\ldots,x_n) \mid x_i \in A_i, 1 \leq i \leq n\}.$$

An n-hyperoperation f on H is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $1 \leq i,j \leq n$ and $x_1^{2n-1} \in H$. We use the notation $f_{(k)}(x_1^{k(n-1)+1})$ to denote $\underbrace{f(f(\ldots f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$, where $k \geq 1$ and $x_1^{k(n-1)+1} \in H$.

H_v-structures were introduced by T. Vougiouklis [32, 33] as a generalization of the well-known algebraic hyperstructures. Some axioms of classical algebraic hyperstructures are replaced by their corresponding weak axioms in H_v-structures. Most of H_v-structures are used in the representation theory. A hypergroupoid $(H, \circ)$ is called an H_v*-semigroup* if $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$. A hypergroupoid $(H, \circ)$ is called a H_v*-group* if it is a quasi-hypergroup and a H_v-semigroup. A multivalued system $(R, +, \cdot)$ is an H_v*-ring* if (1) $(R, +)$ is a H_v-group; (2) $(R, \cdot)$ is a H_v-semigroup; (3) "$\cdot$" is weak distributive with respect to "$+$".

Definition 2.1. A non-empty set M is a H_v-module over a H_v-ring R, if $(M, +)$ is a commutative H_v-group and there exists a map $\star : R \times M \to \mathcal{P}^*(M)$, $(r, x) \to r \star x$ such that

(1) $(r \star (x + y)) \cap (r \star x + r \star y) \neq \emptyset$;
(2) $((r + s) \star x) \cap (r \star x + s \star x) \neq \emptyset$;
(3) $((rs) \star x) \cap (r \star (s \star x)) \neq \emptyset$.

Definition 2.2. An m-ary H_v-semigroup is an algebraic structure (H, f) where

$$\bigcap_{1 \leq i \leq m} f(x_1^{i-1}, f(x_i^{m+i-1}), x_{m+i}^{2m-1}) \neq \emptyset$$

and it is called an m-ary H_v-group if it is m-ary H_v-semigroup and $f(x_1^{i-1}, H, x_{i+1}^m) = H$.

Definition 2.3. An (m,n)-ary H_v-ring is an algebraic structure (R, f, g) which satisfies the following axioms:

(1) (R, f) is an m-ary H_v-group,
(2) (R, g) is an n-ary H_v-semigroup,
(3) The n-ary hyperoperation g is weak distributive with respect to the hyperoperation f, i.e., for every sequence $a_1^{i-1}, a_{i+1}^n, x_1^m$ in R and $1 \leq i \leq n$.

$$g(a_1^{i-1}, (x_1^m), a_{i+1}^n) \cap f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)) \neq \emptyset.$$

Definition 2.4. Let M be a non-empty set. Then (M,h,R,k) is an (m,n)-ary H_v-module over an (m,n)-ary H_v-ring (R,f,g), if (M,h) is a (commutative) m-ary H_v-group and the map

$$k : R^{n-1} \times M \to \mathcal{P}^*(M)$$

satisfies the following conditions:

(1) $k(r_1^{n-1}, h(x_1^m)) \cap h(k(r_1^{n-1}, x_1), \ldots, k(r_1^{n-1}, x_m)) \neq \emptyset$,
(2)

$$k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x) \cap h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)) \neq \emptyset,$$

(3) $k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x) \cap k(r_1^{n-1}, k(r_n^{2n-2}, x)) \neq \emptyset$.

Definition 2.5. Let (M,h_1,R,k_1) and (N,h_2,R,k_2) be (m,n)-ary H_v-modules over an (m,n)-ary H_v-ring (R,f,g). A *strong homomorphism* from M to N is a mapping $\phi : M \to N$ such that for all $a_1, \ldots, a_m, a \in M$ and $r_1, \ldots, r_{n-1} \in R$, the following conditions are satisfied:

(1) $\phi(h_1(a_1, \ldots, a_m)) = h_2(\phi(a_1), \ldots, \phi(a_m))$,
(2) $\phi(k(r_1, \ldots, r_{n-1}, a)) = k(r_1, \ldots, r_{n-1}, \phi(a))$.

ϕ is called an *isomorphism* if it is a bijective strong homomorphism and we write $M \cong N$, and it is called *weak homomorphism* if in (1) and (2) we have non-empty intersection instead of equality.

The notion of P-hyperoperations was introduced for hypergroups (see [29]) and then generalized for H_v-rings (see [8]) and then for H_v-modules (see [12]).

Theorem 2.6. *Let $(H,\cdot)$ be a semigroup, $Z(H)$ be the center of H, $P \subseteq H$, $P \cap Z(H) \neq \emptyset$. Then $(H,P^\star)$ is an n-ary H_v-semigroup; and $(H,P^\star)$ is an n-ary H_v-group if and only if $(H,\cdot)$ is a group. Where the hyperoperation $P^\star$ is defined as follows: For all $x_i \in H$, $i = 1, \ldots, m$, $P^\star(x_1^m) = x_1 \ldots x_m P$.*

3. (m,n)-ary H_v-modules endowed with P-hyperoperations

In this section, we construct (m,n)-ary H_v-modules over (m,n)-ary H_v-rings using P-hyperoperations, study their properties and present some examples.

Let $(R,+,\cdot)$ be a ring, $(M,+,R,\star)$ be an R-module, $P_1, P_2 \subseteq R$ and $P \subseteq M$. We define $P_1^\star, P_2^\star, P^\star, P_\star$ as follows:

For $r_i \in R$, $x_j \in M$ $i \in \{1,\dots,n\}$, and $j \in \{1,\dots,m\}$,

$$\begin{aligned}
P_1^\star(r_1^m) &= r_1 + \ldots + r_m + P_1,\\
P_2^\star(r_1^n) &= r_1 \ldots r_n P_2,\\
P^\star(x_1^m) &= x_1 + \ldots + x_m + P,\\
P_\star(r_1^{n-1}, x) &= r_1 \ldots r_{n-1} P_2 \star x + P.
\end{aligned}$$

In what follows, let $0_R \in P_1$, $P_2 \cap Z(R) \neq \emptyset$ and $0_M \in P$.

Theorem 3.1. *Let R be a ring then $(R, P_1^\star, P_2^\star)$ is an (m,n)-ary H_v-ring.*

Proof. Since $(R,+)$ is a group and $0_R \in P_1 \subseteq R$, it follows, by Theorem 2.6, that $(R, P_1^\star)$ is an m-ary H_v-group. Moreover, having $(R,\cdot)$ a semigroup and the existence of an element $a \in P_2 \cap Z(R)$ implies, by using Theorem 2.6, that $(R, P_2^\star)$ is an n-ary H_v-semigroup. We now show that the weak distributivity is satisfied.

We have

$$P_2^\star(a_1^{i-1}, P_1^\star(x_1^m), a_{i+1}^n) = a_1 \cdots a_{i-1}(x_1 + \ldots + x_m + P_1) a_{i+1} \ldots a_n P_2$$

and

$$\begin{aligned}
&P_1^\star(P_2^\star(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, P_2^\star(a_1^{i-1}, x_m, a_{i+1}^n))\\
&= a_1 \cdots a_{i-1} x_1 a_{i+1} \ldots a_n P_2 + \ldots + a_1 \cdots a_{i-1} x_m a_{i+1} \ldots a_n P_2 + P_1.
\end{aligned}$$

Since $0_R \in P_1$, it follows that there exists $a \in P_2$ such that

$$\begin{aligned}
&a_1 \cdots a_{i-1} x_1 a_{i+1} \ldots a_n a + \ldots x a_1 \cdots a_{i-1} x_m a_{i+1} \ldots a_n a\\
&\in P_2^\star(a_1^{i-1}, P_1^\star(x_1^m), a_{i+1}^n) \cap P_1^\star(P_2^\star(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, P_2^\star(a_1^{i-1}, x_m, a_{i+1}^n)).
\end{aligned}$$

Therefore, $(R, P_1^\star, P_2^\star)$ is an (m,n)-ary H_v-ring. □

Notation 3.2. $(R, P_1^\star, P_2^\star)$ is called an (m,n)**-ary** P**-** H_v**-ring**.

Remark 3.3. For $m = n$, $(R, P_1^\star, P_2^\star)$ is called an n**-ary** P**-** H_v**-ring**. Some examples on n-ary P- H_v-rings are found in [17].

Remark 3.4. For $m = n = 2$, $(R, P_1^\star, P_2^\star)$ is called P**-** H_v**-ring**.

Theorem 3.5. *Let M be an R- module over a ring R such that there exists $a \in P_2 \cap Z(R)$ with $a^2 \in P_2$. Then $(M, P^\star, R, P_\star)$ is an (m,n)-ary H_v-module over $(R, P_1^\star, P_2^\star)$.*

Proof. Since $(M,+)$ is a group and $0_M \in P$, it follows, by Theorem 2.6, that $(M, P^\star)$ is an m-ary H_v-group. We prove that the conditions in Definition 2.4 are satisfied.

(1) We have $P_\star(r_1^{n-1}, P^\star(x_1^m)) = r_1 \ldots r_{n-1} P_2 \star (x_1 + \ldots + x_m + P) + P$ and

$$P^\star(P_\star(r_1^{n-1}, x_1), \ldots, P_\star(r_1^{n-1}, x_m))$$

$$= (r_1 \dots r_{n-1}P_2 \star x_1 + P) + \dots (r_1 \dots r_{n-1}P_2 \star x_m + P) + P.$$

Since $0_M \in P$, it follows that there exists $a \in P_2$ such that

$$r_1 \dots r_{n-1} a \star (x_1 + \dots + x_m)$$
$$\in P_\star(r_1^{n-1}, P^\star(x_1^m)) \cap P^\star(P_\star(r_1^{n-1}, x_1), \dots, P_\star(r_1^{n-1}, x_m)).$$

(2) We have that

$$P_\star(r_1^{i-1}, P_1^\star(s_1^m), r_{i+1}^{n-1}, x)$$
$$= r_1 \dots r_{i-1}(s_1 + \dots + s_m + P_1)r_{i+1} \dots r_{n-1}P_2 \star x + P$$

and

$$P^\star(P_\star(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \dots, P_\star(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))$$
$$= (r_1 \dots r_{i-1}s_1 r_{i+1} \dots r_{n-1}P_2 \star x + P) + \dots$$
$$+(r_1 \dots r_{i-1}s_m r_{i+1} \dots r_{n-1}P_2 \star x + P) + P.$$

Since $0_M \in P$ and $0_R \in P_1$, it follows that there exists $a \in P_2$ such that

$$r_1 \dots r_{i-1}(s_1 + \dots + s_m)r_{i+1} \dots r_{n-1} a \star x$$
$$\in P_\star(r_1^{i-1}, P_1^\star(s_1^m), r_{i+1}^{n-1}, x) \cap P^\star(P_\star(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \dots,$$
$$P_\star(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)).$$

(3) We have that

$$P_\star(r_1^{i-1}, P_2^\star(r_i^{i+n-1}), r_{i+n}^{2n-2}, x)$$
$$= r_1 \dots r_{i-1}(r_i \dots r_{i+n-1}P_2)r_{i+n} \dots r_{2n-2}P_2 \star x + P,$$
$$P_\star(r_1^{n-1}, P_\star(r_n^{2n-2}, x))$$
$$= r_1 \dots r_{n-1}P_2 \star (r_n \dots r_{2n-2}P_2 \star x + P) + P.$$

Since $0_M \in P$ and there exists $a \in P \cap Z(R)$ such that $a^2 \in P$, it follows that

$$r_1 \dots r_{2n-2}a^2 \star x \in P_\star(r_1^{i-1}, P_2^\star(r_1^{i+n-1}), r_{i+n}^{2n-2}, x) \cap P_\star(r_1^{n-1}, P_\star(r_n^{2n-2}, x)).$$

Therefore, $(M, P^\star, R, P_\star)$ is an (m, n)-ary H_v-module. □

Notation 3.6. $(M, P^\star, R, P_\star)$ is called an (m, n)**-ary** P**-** H_v**-module** over $(R, P_1^\star, P_2^\star)$.

Remark 3.7. For $m = n$, $(M, P^\star, R, P_\star)$ is called an n**-ary** P**-** H_v**-module** over the P-H_v-ring $(R, P_1^\star, P_2^\star)$ and for $m = n = 2$, it is called P- H_v-module over R.

We present some examples of (m, n)-ary P-H_v-modules.

Example 3.8. Let M be an R-module and define the following operations: $f : R^m \to R, g : R^n \to R$ as $f(r_1^m) = r_1 + \dots + r_m$ and

$$g(r_1^n) = r_1 \dots r_n,\ h : M^m \to M, k : R^{n-1} \times M \to M$$

as $h(x_1^m) = x_1 + \ldots + x_m$ and $k(r_1^{n-1}, x) = (r_1 \ldots r_{n-1})x$. Then (M, h, R, k) is an (m, n)-ary module over R. If R is a ring with unity "1", we get that (M, h, R, k) is an (m, n)-ary H_v module over R by setting $P = \{0_M\}$, $P_1 = \{0_R\}$ and $P_2 = \{1\}$.

Example 3.9. Let M be any R-module and $P_1 = \{0_R\}$, $P_2 = Z(R)$ and $P = \{0_M\}$. Then $(M, P^\star, R, P_\star)$ is an (m, n)-ary H_v-module (as for every $a \in Z(R)$, $a^2 \in Z(R)$).

Example 3.10. Let $I = \{ar : r \in R\}$ be an ideal of a ring R, $P_1 = \{0, a\} \subseteq R, P_2 = \{0\} \cup \{a^k : k \in \mathbb{N}\} \subseteq R$ and $P = \{0, a^2\}$. Then $(I, P^\star, R, P_\star)$ is an (m, n)-ary H_v-module over $(R, P_1^\star, P_2^\star)$. Moreover, for some particular values, we have $P^\star(a^m) = \{ma, ma + ma^2\}$, $P_\star(a^{n-1}, a) = \{a, a^2, a^{n+k-1} : k \in \mathbb{N}\}$.

Example 3.11. Let $M = \mathbb{Z}_4$ be the group of addition modulo four and $R = \mathbb{Z}$ be the ring of integers with the standard addition and multiplication (M is an R-module). Take $P = \{0, 2\} \subset M$, $P_1, P_2 \subset R$ the set of even integers and the set of odd integers respectively. Theorem 3.1 asserts that $(R, P_1^\star, P_2^\star)$ is an (m, n)-ary P-H_v-ring and Theorem 3.5 asserts that $(M, P^\star, R, P_\star)$ is an (m, n)-ary P-H_v-module over R. By setting $m = n = 3$, we get that M is a 3-ary P-H_v-module over R. The P-hyperoperations in this example are given as follows: For all $r, s, t \in R$, $(R, P_1^\star, P_2^\star)$ is defined as follows:

$$\begin{aligned} P_1^\star(r, s, t) &= \{r + s + t + 2k : k \in \mathbb{Z}\} \\ &= \begin{cases} 2\mathbb{Z}, & \text{if } r + s + t \text{ is an even integer ;} \\ 2\mathbb{Z} + 1, & \text{otherwise.} \end{cases} \\ P_2^\star(r, s, t) &= \{rst(2k + 1) : k \in \mathbb{Z}\}. \end{aligned}$$

The 3-ary H_v-group $(M, P^\star)$ is defined by means of the following tables:

$(0, -, -)$	0	1	2	3
0	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$
1	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$
2	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$
3	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$

$(1, -, -)$	0	1	2	3
0	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$
1	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$
2	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$
3	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$

$(2,-,-)$	0	1	2	3
0	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$
1	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$
2	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$
3	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$

$(3,-,-)$	0	1	2	3
0	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$
1	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$
2	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$
3	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$

and $P_\star$ is defined as follows:

$$P_\star(r,s,x) = \{rs(2k+1)x + \{0,2\}\ mod\ 4 : k \in \mathbb{Z}\}$$
$$= \begin{cases} \{0,2\}, & \text{if } x \equiv 0\ mod\ 4; \\ \{rs(2k+1), 2rs(2k+1) : k \in \mathbb{Z}\} & \text{if } x \equiv 1\ mod\ 4; \\ \{2rs, 2rs+2\}, & \text{if } x \equiv 2\ mod\ 4; \\ \{2rsk+3rs, 2rsk+3rs+2 : k \in \mathbb{Z}\}, & \text{if } x \equiv 3\ mod\ 4. \end{cases}.$$

For some particular values, $P_\star(1,3,2) = \{0,2\}$, $P_1^\star(0,1,2) = P_2$ and $P_2^\star(1,3,1) = \{6k+3 : k \in \mathbb{Z}\}$.

Proposition 3.12. *Let $(M,P^\star,R,P_\star),(N,K^\star,R,K_\star)$ be (m,n)-ary P- H_v-modules over R. Then $M\times N$ is an (m,n)-ary P- H_v-module over R.*

Proof. Let $L = P\times K \subseteq M\times N$. Since $0_{M\times N} \in L$ and $M\times N$ is an R-module, it follows that $(M\times N, L^\star, R, L_\star)$ is an (m,n)-ary P- H_v-module over R. □

Proposition 3.13. *Let $(M,P^\star,R,P_\star),(N,L^\star,R,L_\star)$ be (m,n)-ary P- H_v-modules over R and $\Phi : M \to N$ an R-module isomorphism. Then ϕ is an isomorphism from the (m,n)-ary P- H_v-module M to the (m,n)-ary P- H_v-module N if and only if $\Phi(P) = L$.*

Proof. Let $\phi(P) = L$.
(1) We have that

$$\phi(P^\star(a_1^m)) = \phi(a_1 + \ldots + a_m + P) = \phi(a_1) +' \ldots +' \phi(a_m) +' \phi(P).$$

Since $\phi(P) = L$, it follows that $\phi(P^\star(a_1^m)) = L^\star(\phi(a_1)^m)$.
(2) $\phi(P_\star(r_1^{n-1},x)) = \phi(r_1\ldots r_{n-1}P_2 \star x + P) = r_1\ldots r_{n-1}P_2 \star' \phi(x) +' \phi(P)$. Since $\phi(P) = L$, it follows that $\phi(P_\star(r_1^{n-1},x)) = L_\star(r_1^{n-1},\phi(x))$.

Let $\Phi : M \to N$ be an isomorphism between the (m,n)-ary H_v-modules M

and N. We have that $\phi(0_M) = 0_N$ as ϕ is an R-module isomorphism. Then $\phi(P) = \phi(P^\star(0_M, \dots, 0_M)) = L^\star(0_N, \dots, 0_N) = 0_N + L = L$. □

Proposition 3.14. *Let $\phi : (M, P^\star, R, P_\star) \to (N, L^\star, R, L_\star)$. Then ϕ is an isomorphism if and only if $\phi^{-1} : (N, L^\star, R, L_\star) \to (M, P^\star, R, P_\star)$ is an isomorphism.*

Proof. We prove one direction, the other is done in a similar manner. Let ϕ be isomorphism function. Then $\phi(P^\star(x_1^m)) = L^\star(\phi(x_1)^m)$ and $\phi(P_\star(r_1^{n-1}, x)) = L_\star(r_1^{n-1}, \phi(x))$ for all $x_i, x \in M$ and $r_i \in R$. We get now that $\phi(x_1 + \ldots + x_m + P) = \phi(x_1) + \ldots + \phi(x_m) + L$ and that

$$\phi(r_1 \ldots r_{n-1} P_2 \star x + P) = r_1 \ldots r_{n-1} P_2 \star' \phi(x) +' L.$$

The first implies that $\phi^{-1}(\phi(x_1) + \ldots + \phi(x_m) + L) = x_1 + \ldots + x_m + P$ and that $\phi^{-1}(r_1 \ldots r_{n-1} P_2 \star' \phi(x) +' L) = r_1 \ldots r_{n-1} P_2 \star x + P$. For all $y_i, y \in N$, there exist $x_i, x \in M$ such that $\phi(x_i) = y_i, \phi(x) = y$. We get now that $\phi^{-1}(L^\star(y_1^m)) = \phi^{-1}(y_1 + \ldots + y_m + L)$ and $P^\star(\phi^{-1}(y_1)^m) = x_1 + \ldots + x_m + P$. Moreover,

$$\begin{aligned} \phi^{-1}(L_\star(r_1^{n-1}, y)) &= \phi^{-1}(r_1 \ldots r_{n-1} P_2 \star' y + L) \\ &= r_1 \ldots r_{n-1} P_2 \star x + P \\ &= P_\star(r_1^{n-1}, \phi^{-1}(y)). \end{aligned}$$

□

Proposition 3.15. *Let $\phi, \psi : (M, P^\star, R, P_\star) \to (N, L^\star, R, L_\star)$ be strong homomorphisms and $L + L = L$. Then $\phi + \psi : (M, P^\star, R, P_\star) \to (N, L^\star, R, L_\star)$ is a strong homomorphism.*

Proof. Let $\lambda = \phi + \psi$. We have that

$$\begin{aligned} \lambda(P^\star(x_1^m)) &= \phi(P^\star(x_1^m)) + \psi(P^\star(x_1^m)) \\ &= L^\star(\phi(x_1)^m) + L^\star(\psi(x_1)^m) \\ &= \phi(x_1) + \ldots + \phi(x_m) + L + \psi(x_1) + \ldots + \psi(x_m) + L \\ &= L^\star(\lambda(x_1)^m) \end{aligned}$$

as $L + L = L$. Moreover,

$$\begin{aligned} \lambda(P_\star(r_1^{n-1}, x)) &= \phi(P_\star(r_1^{n-1}, x)) + \psi(P_\star(r_1^{n-1}, x)) \\ &= L_\star(r_1^{n-1}, \phi(x)) + L_\star(r_1^{n-1}, \psi(x)) \\ &= L_\star(r_1^{n-1}, \lambda(x)). \end{aligned}$$

□

Proposition 3.16. *Let $\phi : (M, P^\star, R, P_\star) \to (N, L^\star, R, L_\star)$ be strong homomorphisms such that $\alpha \in R$. If $\alpha \star L = L$, then*
$$\alpha \star \phi : (M, P^\star, R, P_\star) \to (N, L^\star, R, L_\star)$$
is a strong homomorphism.

Proof. Let $\lambda = \alpha \star \phi$. (1) We have that
$$\begin{aligned}\lambda(P^\star(x_1^m)) &= \alpha \star \phi(P^\star(x_1^m))\\ &= \alpha \star L^\star(\phi(x_1)^m)\\ &= \alpha \star (\phi(x_1) + \ldots + \phi(x_m) + L)\\ &= \lambda(x_1) + \ldots \lambda(x_m) + \alpha \star L.\end{aligned}$$
Having $\alpha \star L = L$, implies that $\lambda(P^\star(x_1^m)) = L^\star(\lambda(x_1)^m)$.
(2)
$$\begin{aligned}\lambda(P_\star(r_1^{n-1}, x)) &= \alpha \star (L_\star(r_1^{n-1}, \phi(x))\\ &= \alpha r_1 \ldots r_{n-1} P_2 \star \phi(x) + \alpha \star L\\ &= r_1 \ldots r_{n-1} P_2 \star (\alpha \star \phi(x)) + L\\ &= L_\star(r_1^{n-1}, \lambda(x)).\end{aligned}$$
□

Proposition 3.17. *Let $\phi : (M, P^\star, R, P_\star) \to (N, K^\star, R, K_\star)$ and*
$$\psi : (N, K^\star, R, K_\star) \to (S, L^\star, R, L_\star)$$
be strong homomorphisms. Then $\psi \circ \phi : (M, P^\star, R, P_\star) \to (S, L^\star, R, L_\star)$ is a strong homomorphism.

Proof. The proof is straightforward. □

Proposition 3.18. *Let $(M, P^\star, R, P_\star)$ be an (m, n)-ary P- H_v-module over R, $\alpha \in Z(R)$ and $\phi_\alpha : (M, P^\star, R, P_\star) \to (M, (\alpha P)^\star, R, (\alpha P)_\star)$ such that $\phi_\alpha(x) = \alpha \star x$ for all $x \in M$. Then ϕ_α is a strong homomorphism. Moreover, if R is a ring with unity and α is a unit in R then $(M, (\alpha P)^\star, R, (\alpha P)_\star)$ and $(M, P^\star, R, P_\star)$ are isomorphic (m, n)-ary P- H_v-modules over R.*

Proof. It is clear that ϕ_α is well defined.
(1) $\phi_\alpha(P^\star(x_1^m)) = \phi_\alpha(x_1 + \ldots x_m + P) = \alpha \star (x_1 + \ldots x_m + P)$.
On the other hand,
$$(\alpha P)^\star(\phi_\alpha(x_1), \ldots, \phi_\alpha(x_m)) = \alpha \star x_1 + \ldots + \alpha \star x_m + \alpha \star P.$$
Thus, $\phi_\alpha(P^\star(x_1^m)) = (\alpha P)^\star(\phi_\alpha(x_1), \ldots, \phi_\alpha(x_m))$. (2) We have that

$$\phi_\alpha(P_\star(r_1^{n-1},x)) = \phi_\alpha(r_1\dots r_{n-1}P_2\star x+P) = \alpha r_1\dots r_{n-1}P_2\star x+\alpha P.$$

On the other hand, $(\alpha P)_\star(r_1^{n-1},\phi_\alpha(x)) = r_1\dots r_{n-1}\alpha P_2\star x+\alpha\star P$. Since $\alpha\in Z(R)$, it follows that $\phi_\alpha(P_\star(r_1^{n-1},x)) = (\alpha P)_\star(r_1^{n-1},\phi_\alpha(x))$. Therefore, ϕ_α is strong homomorphism.

If α is a unit in R then $\alpha^{-1}\in R$. The latter implies that ϕ_α is a bijective function. □

Here after, $1\in P_2$ and R is a commutative ring.

Proposition 3.19. *If N is an (m,n)-ary H_v-submodule of $(M,P^\star,R,P_\star)$, then $0_M\in N$, $P\subseteq N$, and moreover $-x\in N$ for all $x\in N$.*

Proof. Since N is an (m,n)-ary H_v-submodule of $(M,P^\star,R,P_\star)$, it follows that $P^\star(0_R,\dots,0_R,x)\subseteq N$ for all $x\in N$. The latter implies that $P\subseteq N$ and hence, $0_M\in N$. Moreover, having $1\in R$ implies that $-1\in R$. For all $x\in N$, we have that $-x=-1(1)\star x+0\in -1P_2\star x+P = P_\star(-1,1,\dots,1,x)\subseteq N$. □

Theorem 3.20. *Let $(M,P^\star,R,P_\star)$ be an (m,n)-ary P- H_v-module. Then N is an (m,n)-ary H_v-submodule of M if and only if $P\subseteq N$ and N is an R-submodule of M.*

Proof. Let N be an R-submodule of M. Then $x+N=N$ and $r\star x\in N$ for all $x\in N$ and $r\in R$. We have that

$$P^\star(x_1^{i-1},N,x_{i+1}^m) = x_1+\ldots+x_{i-1}+N+x_{i+1}+\cdots+x_m+P\subseteq N.$$

Moreover, for every $y\in N$,

$$\begin{aligned} y &= x_1+\ldots+x_{i-1}+(y-x_1-\ldots-x_m)+x_{i+1}+\cdots+x_m+0_M\\ &\in P^\star(x_1^{i-1},N,x_{i+1}^m).\end{aligned}$$

We have that $P_\star(r^{n-1},x)=r_1\dots r_{n-1}P_2\star x+P\subseteq N$. Thus, N is an (m,n)-ary H_v-submodule of M.

Let N be an (m,n)-ary H_v-submodule of M. Proposition 3.19 asserts that $P\subseteq N$. Since $P_\star(0_M,\dots,0_M,N,0_M,\dots,0_M)=N$, it follows that $P+N=N$. For all $x\in N$, $r\in R$,

$$x+N = x+N+P = P^\star(0_M,\dots,0_M,N,x)=N$$

and $r\star x\in P^\star(r,1,\dots,1,x)\subseteq N$. □

Corollary 3.21. *Let $(M,P^\star,R,P_\star)$ be an (m,n)-ary P- H_v-module and N be an (m,n)-ary H_v-submodule of M. Then $|N|\geq|P|$ is a divisor of $|M|$.*

Proof. The proof follows from Theorem 3.20. □

Example 3.22. The 3-ary H_v-module defined in Example 3.11 has only two H_v-submodules: P and M.

Let N be an (m,n)-ary H_v-submodule of M. Then, by Theorem 3.20, N is a submodule of M containing P. Thus, $P+N=N$. We can define M/N as follows:

$$M/N = \{x+N+P : x \in M\} = \{x+N : x \in M\}.$$

We define

$$h^\star : \underbrace{M/N \times \ldots \times M/N}_{m \text{ times}} \to \mathcal{P}^*(M/N) \ , \ h_\star : R^{n-1} \times M/N \to \mathcal{P}^*(M/N)$$

as follows:

$$h^\star(x_1+N,\ldots,x_m+N) = \{x_1+\ldots+x_m+N\}$$

and

$$h_\star(r_1^{n-1}, x+N) = (r_1 \ldots r_{n-1})P_2 \star x + N.$$

Theorem 3.23. *$(M/N, h^\star, R, h_\star)$ is an (m,n)-ary H_v-module over R.*

Proof. It is clear that $(M/N, h^\star)$ is an (m,n)-ary H_v-group. (In particular, it is an (m,n)-ary group.) We need to show that the conditions of Definition 2.4 are satisfied.

(1) We have that

$$h_\star(r_1^{n-1}, h^\star(x_1+N,\ldots,x_m+N)) = r_1 \ldots r_{n-1}P_2 \star (x_1+\ldots+x_m)+N$$

and that

$$\begin{aligned}&h^\star(h_\star(r_1^{n-1},x_1+N),\ldots,h_\star(r_1^{n-1},x_m+N))\\&= r_1\ldots r_{n-1}P_2\star x_1+\ldots+r_1\ldots r_{n-1}P_2\star x_m+N.\end{aligned}$$

It is clear that

$$\begin{aligned}&h_\star(r_1^{n-1}, h^\star(x_1+N,\ldots,x_m+N))\\&\subseteq h^\star(h_\star(r_1^{n-1},x_1+N),\ldots,h_\star(r_1^{n-1},x_m+N)).\end{aligned}$$

(2) We have that

$$h_\star(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x+N) \tag{3.1}$$
$$= r_1\ldots r_{i-1}(s_1+\ldots+s_m)r_{i+1}\ldots r_{n-1}P_2\star x+N \tag{3.2}$$

and that

$$\begin{aligned}&h^\star(h_\star(r_1^{i-1},s_1,r_{i+1}^{n-1},x+N),\ldots,h_\star(r_1^{i-1},s_m,r_{i+1}^{n-1},x+N))\\&= r_1\ldots r_{i-1}s_1r_{i+1}\ldots r_{n-1}P_2\star x+\ldots+r_1\ldots r_{i-1}s_mr_{i+1}\ldots r_{n-1}P_2\star x+N.\end{aligned}$$

It is clear that

$$\begin{aligned}&h_\star(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x+N)\\&\subseteq h^\star(h_\star(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x+N), \ldots, h_\star(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x+N)).\end{aligned}$$

(3) We have that

$$h_\star(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x+N) = r_1 \ldots r_{i+n-1} P_2 r_{i+n} \ldots r_{2n-2} P_2 \star x + N$$

and that

$$h_\star(r_1^{n-1}, h_\star(r_n^{2n-2}, x+N)) = r_1 \ldots r_{n-1} P_2 r_n \ldots r_{2n-2} P_2 \star x + N.$$

Since $1 \in P_2$, it follows that

$$\begin{aligned}&r_1 \ldots r_n \ldots r_{2n-2} \star x + N\\&\in h_\star(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x+N) \cap h_\star(r_1^{n-1}, h_\star(r_n^{2n-2}, x+N)).\end{aligned}$$

This completes the proof. □

Remark 3.24. Let M be an R-module and N_i a submodule of M for $i \in I = \{1, \ldots, k\}$. Then $\bigcap_{i \in J} N_i$ and $\sum_{i \in J} N_i$ are submodules of M for all $J(\neq \emptyset) \subseteq I$.

Proposition 3.25. *Let $(M, P^\star, R, P_\star)$ be an (m,n)-ary P- H_v-module and N_i be an (m,n)-ary H_v-submodule of M for $i \in I = \{1, \ldots, k\}$ and $J \subseteq I$. Then*

1. *$\bigcap\limits_{i \in J} N_i$ is an (m,n)-ary H_v-submodule of M;*
2. *$\sum\limits_{i \in J} N_i$ is an (m,n)-ary H_v-submodule of M.*

Proof. Since N_i is an (m,n)-ary H_v-submodule of M, it follows, by Theorem 3.20, that $P \subseteq N_i$ and that N_i is a submodule of M. The latter and Remark 3.24 imply that both: $\bigcap N_i$ and S are submodules of M containing P. Applying Theorem 3.20, we get our result. □

Remark 3.26. The union of (m,n)-ary H_v-submodules of M is not necessary an (m,n)-ary H_v-submodule of M. The latter follows from Theorem 3.20 and the fact that the union of submodules of an R-module M is not necessary a submodule of M.

4. Isomorphism theorems for (m,n)-ary P-H_v-modules

In this section, we consider (m,n)-ary P-H_v-modules and prove their isomorphism theorems.

Remark 4.1. Let $(M, P_\star, R, P^\star)$ be an (m,n)-ary P-H_v-module over an (m,n)-ary P-H_v-ring $(R, P_1^\star, P_2^\star)$ and S, N are (m,n)-ary H_v-submodules of M. If $N \subseteq S$ then N is an (m,n)-ary H_v-submodule of S.

Theorem 4.2. *First Isomorphism Theorem (Weak form).* *Let $(M, P_\star, R, P^\star)$ and $(N, L_\star, R, L^\star)$ be two (m,n)-ary P-H_v-modules over an (m,n)-ary P-H_v-ring $(R, P_1^\star, P_2^\star)$. If $f : M \to N$ is an R-module homomorphism, $P \subseteq Ker(f), L \subseteq Im(f)$ then (1) $Ker(f)$ is an (m,n)-ary H_v-submodule of M; (2) $Im(f)$ is an (m,n)-ary H_v-submodule of N; (3) There exists a bijective (m,n)-ary R-H_v-module homomorphism $\phi : M/Ker(f) \to Im(f)$.*

Proof. The proofs of (1) and (2) follow from having $Ker(f)$ (containing P) and $Im(f)$ (containing L) R-submodules of M and N respectively (Theorem 3.20). (3) We have that $M/ker(f) = \{x + Ker(f) : x \in M\}$ and $(M/Ker(f), h^\star, R, h_\star)$ an (m,n)-ary H_v-module over R. We define the bijective map $\phi : M/Ker(f) \to Im(f)$ as $\phi(x + Ker(f)) = f(x)$. It is clear that f is a well defined function. We have that

$$\begin{aligned}\phi(h^\star(x_1 + Ker(f), \ldots, x_m + Ker(f))) &= \phi(x_1 + \ldots + x_m + Ker(f)) \\ &= f(x_1 + \ldots + x_m).\end{aligned}$$

Having f an R-module isomorphism implies that

$$\phi(h^\star(x_1 + Ker(f), \ldots, x_m + Ker(f))) = f(x_1) + \ldots + f(x_m).$$

On the other hand,

$$L^\star(\phi(x_1 + Ker(f)), \ldots, \phi(x_m + Ker(f)) = f(x_1) + \ldots + f(x_m) + L.$$

Since $0_N \in L$, it follows that

$$\begin{aligned}&\phi(h^\star(x_1 + Ker(f), \ldots, x_m + Ker(f))) \\ &\subseteq L^\star(\phi(x_1 + Ker(f)), \ldots, \phi(x_m + Ker(f))).\end{aligned}$$

Moreover,

$$\begin{aligned}\phi(h_\star(r_1^{n-1}, x + Ker(f))) &= \phi(r_1 \ldots r_{n-1} P_2 \star x + Ker(f)) \\ &= f(r_1 \ldots r_{n-1} P_2 \star x).\end{aligned}$$

Having f an R-module isomorphism implies that

$$\begin{aligned}\phi(h_\star(r_1^{n-1}, x + Ker(f))) &= r_1 \ldots r_{n-1} P_2 \star' f(x) \\ &\subseteq r_1 \ldots r_{n-1} P_2 \star' \phi(x + Ker(f)) + L \\ &= L^\star(r_1^{n-1}, \phi(x + Ker(f)).\end{aligned}$$

□

Theorem 4.3. ***Second Isomorphism Theorem.*** *Let $(M, P_\star, R, P^\star)$ be an (m,n)-ary P-H_v-module over an (m,n)-ary P-H_v-ring $(R, P_1^\star, P_2^\star)$, S, N are (m,n)-ary H_v-submodules of M. Then $(S+N)/N \cong S/(S\cap N)$.*

Proof. Theorem 3.23 asserts $((S+N)/N, h^\star, R, h_\star)$ and $(S/(S\cap N), k^\star, R, k_\star)$ are (m,n)-ary H_v-modules over R. We define $\phi : (S+N)/N \to S/(S\cap N)$ as $\phi(s+N) = s + (S\cap N)$. We need to prove that ϕ is an R-H_v-module isomorphism. Let $x, y \in S+N$ and $r \in R$.

- ϕ is well defined. Let $x = y$. Then there exist $s, s' \in S$ such that $x = s+N, y = s'+N$. We get, using Proposition 3.19, that $s - s' \in N$ and having $s - s' \in S$ implies that $s - s' \in S\cap N$. Thus, $\phi(x) = \phi(y)$.
- ϕ is one-to-one. Let $\phi(x) = \phi(y)$. Then there exist $s, s' \in S$ such that $x = s+N = y = s'+N$. We get that $s - s' \in S\cap N \subseteq N$ which implies that $s - s' \in N$. Thus, $x = y$.
- ϕ is onto. This clear.
- ϕ is an (m,n)-ary R-H_v-module strong homomorphism. We have that $\phi(h^\star(s_1+N,\ldots,s_m+N)) = \phi(s_1+\ldots+s_m+N) = s_1+\ldots+s_m+S\cap N$. Thus, $\phi(h^\star(s_1+N,\ldots,s_m+N)) = k^\star(\phi(s_1+N),\ldots,\phi(s_m+N))$. Moreover, $\phi(h_\star(r_1^{n-1}, s+N)) = \phi(r_1\ldots r_{n-1}P_2\star s+N) = r_1\ldots r_{n-1}P_2\star s + S\cap N = k_\star(r_1^{n-1}, \phi(s+N))$.

Therefore, $(S+N)/N \cong S/(S\cap N)$. □

Theorem 4.4. ***Third Isomorphism Theorem.*** *Let $(M, P^\star, R, P_\star)$ be an (m,n)-ary P-H_v-module over an (m,n)-ary P-H_v-ring $(R, P_1^\star, P_2^\star)$, S, N are (m,n)-ary H_v-submodules of M with $N \subseteq S \subseteq M$. Then (1) S/N is an (m,n)-ary H_v-submodule of M/N; (2) $(M/N)/(S/N) \cong M/S$.*

Proof. (1) We have $(M/N, h^\star, R, h_\star)$ is an (m,n)-ary H_v-module over R. Let $y = s+N \in S/N$. Having $s \in S$ and S an (m,n)-ary H_v-submodule of M implies that for every $s' \in S$, s' can be written as $s' = s + (s' - s)$. First, we need to show that $h^\star(s_1+N,\ldots,s_{i-1}+N, S/N, s_{i+1}+N,\ldots,s_m+N) = S/N$. It is clear that $h^\star(s_1+N,\ldots,s_{i-1}+N, S/N, s_{i+1}+N, s_m+N) \subseteq S/N$. Let $y = s'+N \in S/N$. Then

$$\begin{aligned} y &\in h^\star(s_1+N,\ldots,s_{i-1}+N, s' - (s_1+\ldots+s_m), s_{i+1}+N,\ldots,s_m+N) \\ &\subseteq h^\star(s_1+N,\ldots,s_{i-1}+N, S/N, s_{i+1}+N,\ldots,s_m+N). \end{aligned}$$

We have that $h_\star(r_1^{n-1}, s+N) = r_1 \dots r_{n-1}P_2 \star s + N \subseteq S/N$.

(2) Since S and S/N are (m,n)-ary H_v-submodules of M and M/N respectively, it follows that $(M/S, k^\star, R, k_\star)$ and $((M/N)/(S/N), l^\star, R, l_\star)$ are (m,n)-ary H_v-modules over R. We have

$$(M/N)/(S/N) = \{x + N \oplus S/N : x \in M\}$$

and $M/S = \{x + S : x \in M\}$. Let $\Phi : (M/N)/(S/N) \to M/S$ with $\Phi(x + N \oplus S/N) = x + S$.

- Φ is well defined. Let $w_1 = x + N \oplus S/N = w_2 = y + N \oplus S/N$. Then $x - y + N \in S/N$. We get that $x - y \in S$ and hence, $x + S = y + S$. Thus, $\Phi(x) = \Phi(y)$.
- Φ is one-to-one. Let $\Phi(w_1) = \Phi(w_2)$. Then $x + S = y + S$ and hence, $x - y \in S$. We get now that $x - y + N \in S/N$. The latter implies that $x + N \oplus S/N = y + N \oplus S/N$. Thus, $w_1 = w_2$.
- Φ is onto. This clear.
- Φ is (m,n)-ary H_v-module strong homomorphism. We have that

$$\begin{aligned}\Phi(l^\star(x_1 + N \oplus S/N, \dots, x_m + N \oplus S/N)) &= \Phi(x_1 + \dots + x_m + N \oplus S/N) \\ &= x_1 + \dots + x_m + S.\end{aligned}$$

Thus,

$$\begin{aligned}&\Phi(l^\star(x_1 + N \oplus S/N, \dots, x_m + N \oplus S/N)) \\ &= k^\star(\Phi(x_1 + N \oplus S/N), \dots, \Phi(x_m + N \oplus S/N))).\end{aligned}$$

Moreover,

$$\begin{aligned}\Phi(l_\star(r_1^{n-1}, x + N \oplus S/N)) &= \Phi(r_1 \dots r_{n-1}P_2 \star' x + N \oplus S/N) \\ &= r_1 \dots r_{n-1}P_2 \star' x + S.\end{aligned}$$

The latter expression is equal to $k_\star(r_1^{n-1}, \Phi(x + N \oplus S/N))$.

Therefore, $(M/N)/(S/N) \cong M/S$. □

5. Conclusion

This paper dealt with a special type of (m,n)-ary H_v-modules that is known as (m,n)-ary P-H_v-modules. Several results related to (m,n)-ary P-H_v-modules, their (m,n)-ary H_v-submodules and the quotient (m,n)-ary H_v-modules were obtained. Also, the Isomorphism theorems for these (m,n)-ary H_v-modules were proved.

For future research, it will be interesting to find a link between (m,n)-ary P-H_v-modules and (m,n)-ary modules by studying their fundamental relation.

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ON (M, N)-ARY P-H_V-MODULES AND THEIR ISOMORPHISM THEOREMS

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بررسی P-H_v- مدول های (m, n) -تایی و قضایای یکریختی آن‌ها

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پس از ارائه تعریف ابرگروه‌ها توسط مارتی، مطالعه ابرساختارها و تعمیم آن‌ها به ابرساختارهای (m, n)-تایی از اهمیت بالایی برخوردار بوده است. در این مقاله، H_v-مدول‌های (m, n)-تایی را روی H_v-حلقه‌ها با استفاده از P- ابر عمل‌ها بنا می‌کنیم. خواص آن‌ها را مطالعه کرده و قضایای یکریختی آن‌ها را اثبات می‌کنیم.

کلمات کلیدی: ابرعمل، H_v-مدول (m, n)-تایی، P-ابرعمل.