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FUZZY PERSPECTIVITY IN FUZZY LATTICES

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Abstract. In this paper, we have introduced and studied the notion of perspectivity in fuzzy lattices. The motivation is from the work done by Wasadikar and Khubchandani. We have tried to relate $\bigtriangledown_F$-relation with fuzzy perspective relation. Also, we prove that for a pair of a fuzzy atoms, the concept of fuzzy subperspective holds. Subsequently, several related properties are proven.

1. Introduction

One of the very important notions in lattice theory is perspectivity and subperspectivity. Janowitz [4] introduced the notion of perspectivity and subperspectivity in a lattice theory. Several researchers have developed many related concepts. In 1965, Zadeh[11] introduced Fuzzy sets and fuzzy relations. In 1971, Zadeh [12] defined fuzzy binary relation and a fuzzy partial order relation. Ajmal and Thomas [1] and Chon [2] defined Fuzzy lattices. Recently, Wasadikar and Khubchandani [7] defined $\bigtriangledown_F$-relation in fuzzy lattices. The motivation is from the work of Maeda and Maeda [5].

In this paper, we define fuzzy perspective and fuzzy subperspective in a fuzzy lattice. Also, we study the relationship between fuzzy subperspective relation and del-relation in fuzzy lattices. We also prove that for a pair of fuzzy atoms, the concept of fuzzy subperspective also holds.

2. Preliminaries

Zadeh [12] introduced the concept of a fuzzy binary relation and a fuzzy partial order relation.

Throughout this paper, X denotes a nonempty set. We recall some concepts.

Definition 2.1. (Chon [2, Definition 2.1]) A mapping $A : X \times X \to [0, 1]$ is called a fuzzy binary relation on X.

Definition 2.2. (Chon [2, Definition 2.1]) A fuzzy binary relation A on X is called:

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(i) fuzzy reflexive: if $A(a,a)=1$, for all $a \in X$;
(ii) fuzzy symmetric: if $A(a,b)=A(b,a)$, for all $a,b \in X$;
(iii) fuzzy transitive: if $A(a,c) \geq \sup_{b \in X} \min[A(a,b),\ A(b,c)]$;
(iv) fuzzy antisymmetric: if $A(a,b)>0$ and $A(b,a)>0$ implies $a=b$.

Definition 2.3. (Chon [2, Definition 2.1]) Let A be a fuzzy binary relation on X.

(i) A is called a fuzzy equivalence relation on X, if A is fuzzy reflexive, fuzzy symmetric and fuzzy transitive.
(ii) A is called a fuzzy partial order relation, if A is fuzzy reflexive, fuzzy antisymmetric and fuzzy transitive. The pair (X,A) is called a fuzzy partially ordered set or a fuzzy poset.
(iii) A is called a fuzzy total order relation, if it is a fuzzy partial order relation and $A(a,b)>0$ or $A(b,a)>0$, for all $a,b \in X$. In this case, the fuzzy poset (X,A) is called a fuzzy totally ordered set or a fuzzy chain.

Several researchers have studied fuzzy lattices. Chon [2] and some others use the terms *upper bound, lower bound* and use the notations $a \vee b$ and $a \wedge b$ to denote the supremum and the infimum of two elements a,b in a fuzzy lattice X in the fuzzy sense. Since the set X is arbitrary, this gives the impression that X itself is a partially ordered set or a lattice. So we use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a,b \in X$, respectively.

Definition 2.4. (Chon [2, Definition 3.1]) Let (X,A) be a fuzzy poset and let $Y \subseteq X$. An element $b \in X$ is said to be a fuzzy upper bound for Y iff $A(a,b)>0$ for all $a \in Y$. A fuzzy upper bound b_0 for Y is called a least upper bound (or supremum) of Y iff $A(b_0,b)>0$ for every fuzzy upper bound b for Y. We then write $b_0 = \sup_F Y = \vee_F Y$. If $Y=\{a,b\}$, then we write $\vee_F Y = a \vee_F b$.

Similarly, an element $c \in X$ is said to be a fuzzy lower bound for Y iff $A(c,a)>0$, for all $a \in Y$. A fuzzy lower bound c_0 for Y is called a fuzzy greatest lower bound (or infimum) of Y iff $A(c,c_0)>0$ for every fuzzy lower bound c for Y. We then write $c_0 = \inf_F Y = \wedge_F Y$. If $Y=\{a,b\}$, then we write $\wedge_F Y = a \wedge_F b$.

Definition 2.5. (Chon [2, Definition 3.2]) Let (X,A) be a fuzzy poset. Then, (X,A) is called a fuzzy lattice if and only if $a \vee_F b$ and $a \wedge_F b$ exist, for all $a,b \in X$.

Definition 2.6. (Mezzomo et. al. [6, Definition 3.4]) A fuzzy lattice (X, A) is said to be bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for every $a \in X$. In this case, $\perp$ and $\top$ are respectively, called bottom and top elements of X.

In the following example, we illustrate these concepts.

Example 2.7. Let $X = \{a, b, c, d, e\}$ and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:
$A(a, a) = A(b, b) = A(c, c) = A(d, d) = A(e, e) = 1.0,$
$A(a, b) = 0.40,\ A(a, c) = 0.50,\ A(a, d) = 0.80,\ A(a, e) = 0.94,$
$A(b, a) = 0.0,\ A(b, c) = 0.0,\ A(b, d) = 0.60,\ A(b, e) = 0.90,$
$A(c, a) = 0.0,\ A(c, b) = 0.0,\ A(c, d) = 0.0,\ A(c, e) = 0.70,$
$A(d, a) = 0.0,\ A(d, b) = 0.0,\ A(d, c) = 0.0,\ A(d, e) = 0.40,$
$A(e, a) = 0.0,\ A(e, b) = 0.0,\ A(e, c) = 0.0,\ A(e, d) = 0.0.$
Then A is a fuzzy partial order relation.
We note that (X, A) is a fuzzy lattice.
The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	e	d	e
c	c	e	c	e	e
d	d	d	e	d	e
e	e	e	e	e	e

$\wedge_F$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	b	b
c	a	a	c	a	c
d	a	b	a	d	d
e	a	b	c	d	e

This fuzzy relation is shown in the following table:

A	a	b	c	d	e
a	1.0	0.42	0.55	0.83	0.94
b	0.0	1.0	0.0	0.62	0.90
c	0.0	0.0	1.0	0.0	0.70
d	0.0	0.0	0.0	1.0	0.40
e	0.0	0.0	0.0	0.0	1.0

Proposition 2.8. *(Chon [2, Proposition 3.3]) Let (X, A) be a fuzzy lattice. For $a, b, c \in X$, the following statements hold:*
(i) $A(a, b) > 0$ iff $a \wedge_F b = a$;
(ii) If $A(b, c) > 0$, then $A(a \wedge_F b, a \wedge_F c) > 0$ and $A(a \vee_F b, a \vee_F c) > 0$.

Chon [2] has considered fuzzy distributivity and fuzzy modularity in fuzzy lattices.

Definition 2.9. (Chon [2, Definition 3.8]) Let (X, A) be a fuzzy lattice. (X, A) is called a fuzzy distributive lattice, if
$a \wedge_F (b \vee_F c) = (a \wedge_F b) \vee_F (a \wedge_F c)$ and
$a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F (a \vee_F c)$ for all $a, b, c \in X$.

Definition 2.10. (Chon [2, Definition 3.12]) A fuzzy lattice (X, A) is called fuzzy modular if $A(a, c) > 0$ implies $a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F c$, for all $a, b, c \in X$.

Definition 2.11. (Wasadikar and Khubchandani [9, Definition 4.4]) Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. Let $a, b \in X$. Then $b \prec_F a$ (a "fuzzy covers" b) if $0 < A(b, a) < 1$, $A(b, c) > 0$ and $A(c, a) > 0$ imply $c = b$ or $c = a$.

Definition 2.12. (Wasadikar and Khubchandani [9, Definition 3.3]) Let P denote the set of all $a \in X$ such that $\perp \prec_F a$. The elements of P are called fuzzy atoms.

Definition 2.13. (Wasadikar and Khubchandani [10, Definition 5.4]) In a fuzzy lattice $\mathcal{L} = (X,\ A)$ with $\perp$, the following property is called the fuzzy exchange property:
If p and q are fuzzy atoms and if $a \wedge p = \perp$, $A(p,\ a \vee q) > 0$ implies $A(q,\ a \vee p) > 0$ (hence implies $a \vee p = a \vee q$).

Definition 2.14. (Wasadikar and Khubchandani [7, Definition 3.3]) Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. Let $a, b \in X$. We write $a \nabla_F b$ if $(x \vee_F a) \wedge_F b = x \wedge_F b$ for every $x \in X$.

Definition 2.15. (Wasadikar and Khubchandani [10, Definition 5.2]) A fuzzy poset $\mathcal{L} = (X,\ A)$ with a least element $\perp$ is atomistic if every element $x \in X$ is the least upper bound of the set of fuzzy atoms less than or equal to x.

Lemma 2.16. *(Wasadikar and Khubchandani [7, Lemma 3.1]) Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$, the following statements hold for $a, b, a_1, b_1 \in X$.*
(i) If $a \nabla_F b$ holds and a_1, b_1 are such that $A(a_1, a) > 0$ and $A(b_1, b) > 0$, then $a_1 \nabla_F b_1$ holds;
(ii) If $a_1 \nabla_F b$ and $a_2 \nabla_F b$ hold, then $(a_1 \vee_F a_2) \nabla_F b$ holds;
(iii) Y^{∇_F} is an ideal of $\mathcal{L}$ for every subset Y of X.

3. Fuzzy perspectivity and fuzzy subperspectivity

In this section, we define fuzzy perspective and fuzzy subperspective relation in fuzzy lattices (X, A). Also, some lemmas and propositions are derived.

Definition 3.1. A fuzzy lattice (X, A) is called fuzzy section semi-complemented lattice $(FSSC)$ if it satisfies the following condition:
If $a \neq b$ in X, then there exists $c \in X$ such that $c \neq \bot$, $A(c, a) > 0$ and $c \wedge_F b = \bot$.

Definition 3.2. A fuzzy lattice (X, A) is called fuzzy dual section semi-complemented lattice $(FSSC^*)$ if it satisfies the following condition:
If $a \neq b$ in X, then there exists $c \in X$ such that $c \neq \top$, $A(a, c) > 0$ and $c \vee_F b = \top$.

Definition 3.3. Let (X, A) be a fuzzy lattice with $\bot$ and $a, b \in X$. Then a and b are fuzzy perspective and written as $a \sim_x^F b$ (or simply $a \sim^F b$) if the following holds:

$$a \vee_F x = b \vee_F x \text{ and } a \wedge_F x = b \wedge_F x = \bot \text{ for some } x \in X.$$

If a fuzzy lattice has $\bot$, then it is evident that $a \sim_\bot^F a$ for any $a \in X$ and $a \sim^F \bot$ implies $a = \bot$.

Definition 3.4. Let (X, A) be a fuzzy lattice with $\bot$ and $a, b \in X$. Then a is fuzzy subperspective to b if the following condition holds:
$A(a, b \vee_F x) > 0$ and $a \wedge_F x = \bot$, for some $x \in X$.

It is evident that $a \sim^F b_1$ and $A(b_1, b) > 0$, then a is fuzzy subperspective to b and that if a is fuzzy subperspective to b if $A(a_1, a) > 0$, then a_1 is fuzzy subperspective to b.

Example 3.5. Let $X = \{\bot, a, b, c, d, e, f, g, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that
$A(\bot, \bot) = A(a, a) = A(b, b) = A(c, c) = A(d, d) = A(e, e) = 1$
$A(f, f) = A(g, g) = A(\top, \top) = 1$,
$A(\bot, a) = 0.2$, $A(\bot, b) = 0.2$, $A(\bot, c) = 0.2$, $A(\bot, d) = 0.2$,
$A(\bot, e) = 0.2$, $A(\bot, f) = 0.2$, $A(\bot, g) = 0.2$, $A(\bot, \top) = 0.02$,
$A(a, \bot) = 0$, $A(a, b) = 0$, $A(a, c) = 0$, $A(a, d) = 0$, $A(a, e) = 0$, $A(a, f) = 0.3$,
$A(a, g) = 0.3$, $A(a, \top) = 0.01$,
$A(b, \bot) = 0$, $A(b, a) = 0$, $A(b, c) = 0$, $A(b, d) = 0$, $A(b, e) = 0$, $A(b, f) = 0$,
$A(b, g) = 0$, $A(b, \top) = 0.01$,
$A(c, \bot) = 0$, $A(c, a) = 0$, $A(c, b) = 0$, $A(c, d) = 0$, $A(c, e) = 0$, $A(c, f) = 0$,
$A(c, g) = 0$, $A(c, \top) = 0.01$,
$A(d, \bot) = 0$, $A(d, a) = 0$, $A(d, b) = 0$, $A(d, c) = 0.3$, $A(d, e) = 0.3$,
$A(d, f) = 0.3$, $A(d, g) = 0$, $A(d, \top) = 0.01$,
$A(e, \bot) = 0$, $A(e, a) = 0$, $A(e, b) = 0$, $A(e, c) = 0$, $A(e, d) = 0$, $A(e, f) = 0$,

$A(e, g) = 0$, $A(e, \top) = 0.01$,
$A(f, \bot) = 0$, $A(f, a) = 0$, $A(f, b) = 0$, $A(f, c) = 0$, $A(f, d) = 0$, $A(f, e) = 0$, $A(f, g) = 0$, $A(f, \top) = 0.01$,
$A(g, \bot) = 0$, $A(g, a) = 0$, $A(g, b) = 0$, $A(g, c) = 0$, $A(g, d) = 0$, $A(g, e) = 0$, $A(g, f) = 0$, $A(g, \top) = 0.01$,
$A(\top, \bot) = 0$, $A(\top, a) = 0$, $A(\top, b) = 0$, $A(\top, c) = 0$, $A(\top, d) = 0$, $A(\top, e) = 0$, $A(\top, f) = 0$, $A(\top, g) = 0$.

This fuzzy relation is shown in the following table:

A	⊥	a	b	c	d	e	f	g	⊤
⊥	1.0	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
a	0.0	1.0	0.0	0.0	0.0	0.0	0.3	0.3	0.01
b	0.0	0.0	1.0	0.0	0.0	0.0	0.3	0.0	0.01
c	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.01
d	0.0	0.0	0.0	0.3	1.0	0.3	0.3	0.0	0.01
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.01
f	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.01
g	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.01
⊤	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	⊥	a	b	c	d	e	f	g	⊤
⊥	⊥	a	b	c	d	e	f	g	⊤
a	a	a	f	f	f	⊤	f	g	⊤
b	b	f	b	f	f	⊤	f	⊤	⊤
c	c	f	f	c	d	e	f	⊤	⊤
d	d	f	f	d	d	e	f	⊤	⊤
e	e	⊤	⊤	e	e	e	⊤	⊤	⊤
f	f	f	f	f	f	⊤	f	⊤	⊤
g	g	g	⊤	⊤	⊤	⊤	⊤	g	⊤
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤

$\wedge_F$	⊥	a	b	c	d	e	f	g	⊤
⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
a	⊥	a	⊥	⊥	⊥	⊥	a	a	a
b	⊥	⊥	b	⊥	⊥	⊥	b	⊥	b
c	⊥	⊥	⊥	c	c	c	⊥	⊥	c
d	⊥	⊥	⊥	c	d	d	d	⊥	d
e	⊥	⊥	⊥	c	d	e	d	⊥	e
f	⊥	a	b	⊥	d	d	f	a	f
g	⊥	a	⊥	⊥	⊥	⊥	a	g	g
⊤	⊥	a	b	c	d	e	f	g	⊤

Then A is a fuzzy partial order relation.
We note that (X, A) is a fuzzy lattice.
As $a \vee_F b = b \vee_F d = f$ and $a \wedge_F b = b \wedge_F d = \bot$, then a and b are fuzzy perspective, i.e., $a \sim_b^F d$.
Also, as $A(a, b \vee_F d) = A(a, f) = 0.3 > 0$ and $a \wedge_F b = \bot$, then a is fuzzy subperspective to d.

Lemma 3.6. *In a fuzzy lattice with $\bot$, if b is fuzzy subperspective to a and if $a \nabla_F b$, then $b = \bot$. If $a \sim^F b$ and $a \nabla_F b$, then $a = b = \bot$.*

Proof. Suppose that b is fuzzy subperspective to a.
Then $A(b, a \vee_F x) > 0$ and

$$b \wedge_F x = \bot. \tag{3.1}$$

As $a \nabla_F b$ holds we have

$$(x \vee_F a) \wedge_F b = x \wedge_F b. \tag{3.2}$$

As $A(b, a \vee_F x) > 0$ by (i) of Proposition 2.8 we have $b \wedge_F (a \vee_F x) = b$. So, equation (3.2) reduces to $b = x \wedge_F b$. By equation (3.1) we get $b = \bot$. The proof of second statement follows from first statement. □

Lemma 3.7. *Let a and b be elements of a $FSSC$ lattice (X, A). The following statements are equivalent:*

(i) $a \nabla_F b$;
(ii) If $A(b_1, b) > 0$ and b_1 is fuzzy subperspective to a, then $b_1 = \bot$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $A(b_1, b) > 0$ and b_1 is fuzzy subperspective to a, by (i) of Lemma 2.16 we have $a \nabla_F b_1$. By Lemma 3.6 we have $b_1 = \bot$.

$(ii) \Rightarrow (i)$ Assume that $a \nabla_F b$ does not hold. There exists $x \in X$ such that $(x \vee_F a) \wedge_F b \neq x \wedge_F b$.
By Definition 3.1 there exists b_1 such that $b_1 \neq \bot$,
$A(b_1, (x \vee_F a) \wedge_F b) > 0$ and

$$b_1 \wedge_F x \wedge_F b = \bot. \tag{3.3}$$

As $A(b_1, b) > 0$ by (i) of Proposition 2.8 we have $b_1 \wedge_F b = b_1$. So, equation (3.3) reduces to $b_1 \wedge_F x = \bot$. Hence b_1 is fuzzy subperspective to a, which is contradiction to (ii). Hence $a \nabla_F b$ holds. □

Lemma 3.8. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\bot$ and $a \in X$. An fuzzy atom $p \in X$ is fuzzy subperspective to a if and only if $a \nabla_F p$ does not hold.*

Proof. If p is fuzzy subperspective to a, then by Lemma 3.6 $a \nabla_F p$ does not hold.

Conversely, if $a \nabla_F p$ does not hold, then there exists $x \in X$ such that $(x \vee_F a) \wedge_F p \neq x \wedge_F p$. Since $(x \vee_F a) \wedge_F p \neq \bot$ we have $A(p, x \vee_F a) > 0$. Since $p \neq x \wedge_F p$, we have $x \wedge_F p = \bot$. Hence p is fuzzy subperspective to a. □

Lemma 3.9. *Let a and b be elements of an atomistic lattice $\mathcal{L} = (X, A)$. Then the following statements are equivalent:*

(i) $a \nabla_F b$;

(ii) $a\nabla_F p$ *holds for every fuzzy atom* $p \in X$ *with* $A(p,b) > 0$*;*

(iii) There exists no fuzzy atom $p \in X$ *such that* $A(p,b) > 0$ *and* p *is fuzzy subperspective to* a*;*

(iv) There exists no bottom element such that $A(b_1,b) > 0$ *and* b_1 *is fuzzy subperspective to* a*.*

Proof. Since $\mathcal{L} = (X, A)$ is $FSSC$, the equivalence of (i) and (iv) follows from Lemma 3.7.

The equivalence of (ii) and (iii) follows from Lemma 3.8.

The implication (iv) and (iii) is trivial.

We shall prove the implication $(iii) \Rightarrow (i)$. If $(x \vee_F a) \wedge_F b \neq x \wedge_F b$, then there exists a fuzzy atom $p \in X$ such that $A(p,(x \vee_F a) \wedge_F b) > 0$ and $A(p, x \wedge_F b) = 0$, i.e., $p \wedge_F x \wedge_F b = \perp$. Then $A(p, a \vee_F x) > 0$ and since $A(p,b) > 0$. By (i) of Proposition 2.8 we have $p \wedge_F b = p$. By taking meet x on both sides. Therefore we have $p \wedge_F x = p \wedge_F b \wedge_F x = \perp$. This contradicts to (iii). □

Theorem 3.10. *Let* a *and* b *be elements of* $FSSC^*$ *lattice* $\mathcal{L} = (X, A)$ *with* $\perp$*. Then the following statements are equivalent:*

(i) $a\nabla_F b$*;*

(ii) $x \vee_F a = \top$ *implies* $A(b,x) > 0$*;*

(iii) $x = (x \vee_F a) \wedge_F (x \vee_F b)$ *for every* $x \in X$*.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $a\nabla_F b$ holds. So we have

$$(x \vee_F a) \wedge_F b = x \wedge_F b. \tag{3.4}$$

If $x \vee_F a = \top$, then by equation (3.4) we have

$$x \wedge_F b = (x \vee_F a) \wedge_F b = \top \wedge_F b = b$$

i.e., $b = x \wedge_F b$.
Therefore by (i) of Proposition 2.8 we have $A(b,x) > 0$.
$(ii) \Rightarrow (iii)$ For $x \in X$ we put $y = (x \vee_F a) \wedge_F (x \vee_F b)$. By (i) of Proposition 2.8 we get $A(y, x \vee_F a) > 0$. By (ii) of Proposition 2.8 we have

$$A(c \vee_F y, c \vee_F x \vee_F a) > 0. \tag{3.5}$$

If $y \neq x$, then by Definition 3.2 there exists $c \in X$ such that $c \neq \top$, $A(x,c) > 0$ and $c \vee_F y = \top$. Therefore equation (3.5) reduces to $A(\top, c \vee_F a) > 0$. As $A(c \vee_F a, \top) > 0$ always holds. By fuzzy antisymmetry of A we have $c \vee_F a = \top$. Similarly, we can show $c \vee_F b = \top$. By (ii) $c \vee_F a = \top$ implies $A(b,c) > 0$. Hence $c = c \vee_F b = \top$, a contradiction. Therefore we have $x = y = (x \vee_F a) \wedge_F (x \vee_F b)$.

$(iii) \Rightarrow (i)$ For any $x \in X$ by (iii) we have $x = (x \vee_F a) \wedge_F (x \vee_F b)$.
By taking meet b on both sides we get $x \wedge_F b = (x \vee_F a) \wedge_F (x \vee_F b) \wedge_F b$.
By using fuzzy absorption identity we have $x \wedge_F b = (x \vee_F a) \wedge_F b$.
Hence (i) holds. □

Lemma 3.11. *Let $\mathcal{L} = (X, A)$ be a $FSSC^*$ lattice with $\perp$ and let p and q be fuzzy atoms of (X, A). The following statements are equivalent:*
(i) $p \sim^F q$;
(ii) q is fuzzy subperspective of p;
(iii) $p \nabla_F q$ does not hold.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial.

The equivalence of (ii) and (iii) follows from Lemma 3.8.

$(iii) \Rightarrow (i)$ If $p \nabla_F q$ does not hold, then by Theorem 3.10 there exists $x \in X$ such that $x \neq (x \vee_F p) \wedge_F (x \vee_F q)$. By Definition 3.2 there exists $c \in X$ such that $c \neq \top$, $A(x, c) > 0$ and $c \vee_F \{(x \vee_F p) \wedge_F (x \vee_F q)\} = \top$. Then $c \vee_F p = c \vee_F x \vee_F p = \top$ and $c \vee_F q = c \vee_F x \vee_F q = \top$. If $c \wedge_F p \neq \perp$, then $A(p, c) > 0$ hence $c = c \vee_F p = \top$, a contradiction. Hence $c \wedge_F p = \perp$. Similarly, $c \wedge_F q = \perp$. Therefore $p \sim_c^F q$. □

Lemma 3.12. *Let p and q be fuzzy atoms of a fuzzy lattice with exchange property. The following statements are equivalent:*
(i) $p \sim^F q$;
(ii) q is fuzzy subperspective of p;
(iii) $p \nabla_F q$ does not hold.

Proof. $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (i)$ If $A(p, q \vee_F x) > 0$ and $p \wedge_F x = \perp$, then we have $q \wedge_F x = \perp$ for otherwise $A(p, q \vee_F x) > 0$, i.e., $A(p, x) > 0$, a contradiction to $p \wedge_F x = \perp$. Moreover, by the exchange property we have $p \vee_F x = q \vee_F x$. Therefore we have $p \vee_F x = q \vee_F x$ and $p \wedge_F x = q \wedge_F x = \perp$. Hence $p \sim_x^F q$.

The equivalence of (ii) and (iii) follows from Lemma 3.8. □

4. Conclusion

In this paper, we have introduced the notion of fuzzy perspective relation in fuzzy lattice. Also, we have obtained some properties.

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FUZZY PERSPECTIVITY IN FUZZY LATTICES

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در این مقاله، ما مفهوم پرسپکتیویته در شبکه‌های فازی را معرفی و بررسی کرده‌ایم. انگیزه این کار از پژوهش انجام‌شده توسط واسادیکار و خوبچندانی الهام گرفته شده است. ما تلاش کرده‌ایم که $\bigtriangledown_F$-رابطه را با رابطه پرسپکتیو فازی مرتبط کنیم. همچنین ثابت می‌کنیم که برای یک جفت از اتم‌های فازی، مفهوم زیرپرسپکتیویته فازی برقرار است. در ادامه، خواص مرتبط متعددی اثبات می‌شوند.

کلمات کلیدی: مشبکه‌های فازی، پرسپکتیویته فازی، زیرپرسپکتیویته فازی، دل-رابطه فازی، اتم فازی، اتمیستی.