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ON WEAK EXTENDED ORDER ALGEBRAS WITH ADJOINT PAIRS AND GALOIS PAIRS

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Abstract. In this paper, we consider properties of weak extended order algebras with adjoint pairs and Galois pairs, and prove some new results. Moreover, we clarify the relation between these algebras and BCK-algebras, that is, the class of all normal weak extended order algebras with adjoint pair satisfying the condition $\top \to x = x$ is identical with the class of all BCK-algebras with the condition (S).

1. Introduction

An algebraic treatment is very useful to consider properties of various logics. In this case, every formula in the logic is interpreted and evaluated in the algebra. For example, a formula $A \to A$ is provable in the classical propositional logic (CPL) by use of axioms and inference rules of the logic. However, this method is not easy for the complicated formulas such as $(A \to (B \to C)) \to (B \to (A \to C))$ on one hand. It needs much effort to get that the formula is provable using axioms and inference rules of CPL. On the other hand, if we use an algebraic method for CPL, then such formulas are interpreted and evaluated in the class of Boolean algebras and proved to be true, that is, the values of the formulas equal to 1 in the Boolean algebras.

A uniform algebraic treatment of various logics requires the choice of suitable algebras and every logical symbol is interpreted in such algebras. Especially, the "implication" logical symbol $\to$ is important, because it plays an essential role to get other provable formulas by axioms and inference rules of the logic. This explains the importance of interpretation of implication symbol as the basic algebraic tool. In [8], an implication logic is introduced and implicative algebra is considered as an algebraic semantics of the implication logic. Since then, many interesting logics such as BCK-logic ([5]) are introduced and studied using algebraic method such as filters [10]. After that many algebras corresponding interesting logics are considered from the view point of algebraic interest ([1, 2, 4, 6, 7, 9]).

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A binary relation R from a set X to a set Y is a subset $R \subseteq X \times Y$ and a binary L-relation from X to Y is a map $R : X \times Y \to L$, where L is usually assumed to be some kind of algebra. For a distinguished element $d \in L$, every L-relation R from X to Y can be considered as a map $X \times Y \to L$ such that $(x, y) \in R$ if and only if $R(x, y) = d$ for all $(x, y) \in X \times Y$. Especially, a partial order R on a set X can be represented as a map $X \times X \to L$ such that for all $x, y, z \in X$

$R(x, x) = d$ (reflexivity);
$R(x, y) = R(y, x) = d \Rightarrow x = y$ (antisymmetry);
$R(x, y) = R(y, z) = d \Rightarrow R(x, z) = d$ (transitivity).

This means that any ($L-$) relation on a set X can be considered as a map from $X \times X$ to L with the distinguished element $d \in L$. Considering the cardinality of $2^{X \times X}$ and $L^{X \times X}$, a notion of maps from $X \times X$ to L is more general than that of relations. Roughly speaking, the class of all maps from $X \times X$ to L covers the class of all relations on X.

In this paper, we consider properties of weak extended order algebras. The notion of weak extended order algebras is introduced in [4] and its basic results are proved in [1, 2, 4]. The algebra $(X, \to, \top)$ of type $(2, 0)$ has $\to$ as a primitive operator and can be considered as another representation of partially ordered set with a greatest element $\top$, which are also called implicative algebras [8]. The binary operation $\to$ represents an extension of the order relation $\leq$ of a partially ordered set X with a greatest element $\top$. Therefore, the class $w\mathcal{EO}$ of all weak extended order algebras is a wide algebraic class which contains the class $\mathcal{BCK}$ of BCK-algebras, the class of Hilbert algebras, and so on, as subclasses. Since weak extended order algebras are embedded into complete such algebras by the MacNeille completion method, a new operator $\odot$ can be defined under the condition of infinite distributivity as follows:

$$x \odot y = \bigwedge \{t \mid y \leq x \to t\}.$$

From the definition, it is clear that the operator $\odot$ is a left adjoint to the operator $\to$,

$$x \odot y \leq z \ \Leftrightarrow \ y \leq x \to z,$$

that is, $(\odot, \to)$ is an adjoint pair. On the other hand, for the primitive operator $\to$, the new operators $\odot$ and $\rightsquigarrow$ are axiomatically introduced as an adjoint pair $(\odot, \to)$ and $(\to, \rightsquigarrow)$ as a Galois pair to weak extended order algebras [6, 7], in which the completeness and the infinite distributivity are not assumed. A new notion of *adjoint triple* $(\to, \odot, \rightsquigarrow)$ is also introduced

$$x \odot y \le z \;\Leftrightarrow\; y \le x \to z \;\Leftrightarrow\; x \le y \rightsquigarrow z$$

and its basic properties are considered in [6, 7]. As proved below, since the notion of adjoint triple $(\to, \odot, \rightsquigarrow)$ is equivalent to the fact that $(\odot, \to)$ is the adjoint pair and $(\to, \rightsquigarrow)$ is the Galois pair, essentially new results are not deduced from the notion of adjoint triple. In addition, a definition of *symmetrical* in [6, 7] is not sufficient. Because, a given symmetrical algebra and its induced algebra are not isomorphic. This looses the meaning of symmetric.

In this paper, we consider in detail properties of weak extended algebras with adjoint pairs (AP) and Galois pairs (GP). We provide essentially new results and simple proofs of the results obtained so far. Moreover, we clarify the relation between these algebras and BCK-algebras [5, 11].

2. (Weak) Extended order algebras

We recall a definition of (weak) extended order algebras [1, 2, 4, 6, 7]. An algebraic structure $(X, \to, \top)$ is called a *weak extended order algebra* (simply *w-eo algebra*) if

(E1) $x \to \top = \top$ for all $x \in X$;
(E2) $x \to x = \top$ for all $x \in X$;
(E3) $x \to y = \top$ and $y \to x = \top \Rightarrow x = y$ for all $x, y \in X$;
(E4) $x \to y = \top$ and $y \to z = \top \Rightarrow x \to z = \top$ for all $x, y, z \in X$.

An *extended order algebra* (*eo-algebra*) is a w-eo algebra satisfying the conditions

(EO1) $x \to y = \top \;\Rightarrow\; (z \to x) \to (z \to y) = \top$;
(EO2) $x \to y = \top \;\Rightarrow\; (y \to z) \to (x \to z) = \top$.

In addition, a w-eo algebra X is called *normal* if it satisfies

(N1) $(x \to y) \to ((z \to x) \to (z \to y)) = \top$;
(N2) $(x \to y) \to ((y \to z) \to (x \to z)) = \top$.

It is clear that every normal w-eo algebra is an eo-algebra.

Example 1. Let $X = \{a, b, c, d, \top\}$ and the operation $\to$ be defined by the following table:

$\to$	a	b	c	d	$\top$
a	$\top$	$\top$	b	a	$\top$
b	b	$\top$	b	a	$\top$
c	b	b	b	$\top$	$\top$
d	b	b	b	$\top$	$\top$
$\top$	b	b	b	a	$\top$

The algebra is an eo-algebra. But this is not normal, because

$$(c \to d) \to ((b \to c) \to (b \to d)) = \top \to (b \to a) = \top \to b = b \neq \top,$$

that is, (N2) does not hold in the w-eo algebra.

We define a binary relation $\leq$ on the w-eo algebra $(X, \to, \top)$ by

$$x \leq y \;\Leftrightarrow\; x \to y = \top,$$

then $(X, \leq, \top)$ satisfies

(E1') $x \leq \top$;
(E2') $x \leq x$;
(E3') $x \leq y,\ y \leq x \;\Rightarrow\; x = y$;
(E4') $x \leq y,\ y \leq z \;\Rightarrow\; x \leq z$,

in other words, X is a partially ordered set (poset) with a top (greatest) element $\top$.

Now we define adjoint pairs , Galois pairs and adjoint triples, which play important roles in the paper.

A w-eo algebra $(X, \to, \top)$ has an *adjoint pair* (AP) $(\to, \odot)$ if there exists a binary operation $\odot$ on X such that for all $x, y, z \in X$,

$$x \odot y \to z = \top \;\Leftrightarrow\; y \to (x \to z) = \top,$$

that is,

$$x \odot y \leq z \;\Leftrightarrow\; y \leq x \to z.$$

Example 2. Let $X = ([0,1], \to, 1)$. We define operators $\odot$ and $\to$ as follows:

$$x \odot y = \min\{x, y\}, \quad x \to y = \begin{cases} 1 & (x \leq y) \\ y & (\text{otherwise}) \end{cases}$$

It is easy to show that X is an eo-algebra and has an adjoint pair (AP) $(\to, \odot)$:

$$x \odot y \leq z \;\Leftrightarrow\; y \leq x \to z.$$

A w-eo algebra $(X, \to, \top)$ has a *Galois pair* (GP) $(\to, \rightsquigarrow)$ if there exists a binary operation $\rightsquigarrow$ on X such that for all $x, y, z \in X$,

$$x \to (y \to z) = \top \;\Leftrightarrow\; y \to (x \rightsquigarrow z) = \top,$$

or equivalently,

$$x \le y \to z \;\Leftrightarrow\; y \le x \rightsquigarrow z.$$

Example 3. Let X be a partially ordered set $\{0, a, b, c, 1\}$, where $0 < a < b, c < 1$ and b, c are incomparable. Consider the operations $\to, \rightsquigarrow$ defined by the following tables:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	1	c	c	1
c	b	b	b	1	1
1	0	a	b	c	1

It is a routine work to show that X is an eo-algebra and has a Galois pair (GP) $(\to, \rightsquigarrow)$:

$$x \le y \to z \;\Leftrightarrow\; y \le x \rightsquigarrow z.$$

A w-eo algebra $(X, \to, \top)$ is said to be *symmetrical* if it has a Galois pair $(\to, \rightsquigarrow)$ and $(X, \rightsquigarrow, \top)$ is a w-eo algebra, moreover, the partial orders induced by the operations $\to$ and $\rightsquigarrow$ are identical. We note that the algebra in the Example 2 is the symmetric eo-algebra.

Remark 2.1. In [6, 7], the notion of symmetrical is defined as follows:

> A w-eo algebra X is called *symmetrical* if it has a Galois pair $(\to, \rightsquigarrow)$ and $(X, \rightsquigarrow, \top)$ is a w-eo algebra.

This definition is different from the original one in [4]. In addition, the definition in [6, 7] is not sufficient, because the two orders $\le$ and $\preceq$ induced by $\to$ and $\rightsquigarrow$ respectively, are not identical in general, that is, $x \to y = \top$ is not equivalent to $x \rightsquigarrow y = \top$. This means that for a symmetrical w-eo algebra $(X, \to, \top)$, its symmetric induced algebra $(X, \rightsquigarrow, \top)$ is not isomorphic to $(X, \to, \top)$ in general. This looses the meaning of symmetric and expected results. Therefore we adopt the original definition provided in [4].

A w-eo algebra $(X, \to, \top)$ has an *adjoint triple* (AT) $(\to, \odot, \rightsquigarrow)$ if there exist binary operations $\odot$ and $\rightsquigarrow$ such that for all $x, y, z \in X$,

$$x \odot y \le z \;\Leftrightarrow\; y \le x \to z \;\Leftrightarrow\; x \le y \rightsquigarrow z.$$

Example 4. Let $X = \{0, a, b, c, 1\}$, where $0 < a < b, c < 1$ and b, c are incomparable. Consider the operations $\odot, \rightarrow, \rightsquigarrow$ defined by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	1	c	c	1
c	b	b	b	1	1
1	0	a	b	c	1

It is easy to show that the algebra X is an eo-algebra with adjoint triple (AT) $(\rightarrow, \odot, \rightsquigarrow)$:

$$x \odot y \leq z \;\Leftrightarrow\; y \leq x \rightarrow z \;\Leftrightarrow\; x \leq y \rightsquigarrow z.$$

We note that a w-eo algebra $(X, \rightarrow, \top)$ has another name *implicative algebra* [8].

An algebra $(X, \rightarrow, \top)$ of type $(2, 0)$ is called a *BCK-algebra* [5] if

(B1) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = \top$;
(B2) $x \rightarrow ((x \rightarrow y) \rightarrow y) = \top$;
(B3) $x \rightarrow x = \top$;
(B4) $x \rightarrow \top = \top$;
(B5) $x \rightarrow y = \top$ and $y \rightarrow x = \top \Rightarrow x = y$.

It is obvious that every BCK-algebra is a w-eo algebra but the converse does not hold in general.

Proposition 2.2. *Let $(X, \rightarrow, \top)$ be a w-eo algebra. If it has a Galois pair (GP) $(\rightarrow, \rightsquigarrow)$ then (EO2) condition holds:*

$$(EO2) \quad x \leq y \;\Rightarrow\; y \rightarrow z \leq x \rightarrow z \quad (\forall x, y, z \in X).$$

Proof. For all $t \in X$, since

$$t \leq y \rightarrow z \Leftrightarrow y \leq t \rightsquigarrow z \Rightarrow x \leq y \leq t \rightsquigarrow z \Rightarrow x \leq t \rightsquigarrow z \Rightarrow t \leq x \rightarrow z,$$

we have $y \rightarrow z \leq x \rightarrow z$. □

Proposition 2.3. *Let $(X, \rightarrow, \top)$ be a w-eo algebra. If it has a Galois pair (GP) $(\rightarrow, \rightsquigarrow)$ then $\top \rightsquigarrow x = x$ for all $x \in X$.*

Proof. Since $\top \leq x \rightarrow x$ and $(\rightarrow, \rightsquigarrow)$ is the Galois pair, we have $x \leq \top \rightsquigarrow x$. Moreover, it follows from $\top \rightsquigarrow x \leq \top \rightsquigarrow x$ that $\top \leq (\top \rightsquigarrow x) \rightarrow x$ and so $(\top \rightsquigarrow x) \rightarrow x = \top$, that is, $\top \rightsquigarrow x \leq x$. Therefore, $\top \rightsquigarrow x = x$. □

Lemma 2.4. *Let* $(X, \to, \top)$ *be a w-eo algebra with (GP)* $(\to, \rightsquigarrow)$. *If* $\top \to x = x$ *for all* $x \in X$, *then* $x \to y = \top$ *if and only if* $x \rightsquigarrow y = \top$ *for all* $x, y \in X$.

Proof. ($\Rightarrow$) Suppose that $x \to y = \top$. Since $y = \top \to y$, we get

$$\top = x \to y = x \to (\top \to y) = \top \to (x \rightsquigarrow y) = x \rightsquigarrow y.$$

($\Leftarrow$) Conversely, let $x \rightsquigarrow y = \top$. We have

$$\top = \top \to \top = \top \to (x \rightsquigarrow y) = x \to (\top \to y) = x \to y.$$

□

The next result is a characterization of symmetrical w-eo algebras.

Theorem 2.5. *Let* $(X, \to, \top)$ *be a w-eo algebra with (GP)* $(\to, \rightsquigarrow)$. *Then we have* $\top \to x = x$ *for all* $x \in X$ *if and only if* $(X, \to, \top)$ *is symmetrical.*

Proof. ($\Rightarrow$) Let $\top \to x = x$ for all $x \in X$. It is sufficient to show that $(X, \rightsquigarrow, \top)$ is a w-eo algebra and satisfies

$$x \to y = \top \;\Leftrightarrow\; x \rightsquigarrow y = \top.$$

This is obvious from Lemma 2.4.

($\Leftarrow$) Conversely, suppose that $(X, \to, \top)$ is a symmetrical w-eo algebra with (GP) $(\to, \rightsquigarrow)$. Since $(X, \rightsquigarrow, \top)$ is the w-eo algebra and $\top = x \rightsquigarrow x$, we have $x \le \top \to x$. On the other hand, for all $u \in X$, if $u \le \top \to x$ then $\top \le u \rightsquigarrow x$ and thus $u \rightsquigarrow x = \top$. Since the partial orders induced by $\to$ and $\rightsquigarrow$ are the same, we get $u \to x = \top$, that is,

$$\forall u \,(u \le \top \to x \;\Rightarrow\; u \le x).$$

Therefore, $\top \to x \le x$ and thus $\top \to x = x$ for all $x \in X$. □

A w-eo algebra $(X, \to, \top)$ is called *commutative* if it satisfies

$$x \to (y \to z) = \top \;\Leftrightarrow\; y \to (x \to z) = \top \quad (\forall x, y, z \in X).$$

If a w-eo algebra has (GP), then the commutativity relates $\to$ and $\rightsquigarrow$ as follows.

Proposition 2.6. *For any w-eo algebra* $(X, \to, \top)$ *with (GP), it is commutative if and only if* $x \to y = x \rightsquigarrow y$ *for all* $x, y \in X$.

Proof. ($\Rightarrow$) For all $u \in X$, since

$$\begin{aligned} u \le x \to y \;&\Leftrightarrow\; u \to (x \to y) = \top \\ &\Leftrightarrow\; x \to (u \to y) = \top \quad \text{(by commutativity)} \end{aligned}$$

$$\Leftrightarrow\ x \le u \to y\ \Leftrightarrow\ u \le x \rightsquigarrow y \quad \text{(by (GP))},$$

we obtain $x \to y = x \rightsquigarrow y$.

($\Leftarrow$) Conversely, since

$$\begin{aligned} x \to (y \to z) = \top\ &\Leftrightarrow\ x \le y \to z = y \rightsquigarrow z \\ &\Leftrightarrow\ y \le x \to z \\ &\Leftrightarrow\ y \to (x \to z) = \top, \end{aligned}$$

X is commutative.

□

3. w-eo algebras with (AP) $(\odot, \to)$

In this section we consider properties of w-eo algebras with (AP) $(\odot, \to)$ in detail and provide new results. Let $(X, \to \top)$ be a w-eo algebra with (AP) $(\odot, \to)$, that is,

$$x \odot y \le z\ \Leftrightarrow\ y \le x \to z \quad (\forall x, y, z \in X).$$

The following is a basic result for w-eo algebras with (AP) $(\odot, \to)$.

Proposition 3.1. *For a w-eo algebra X with (AP) $(\odot, \to)$,*
(1) *(EO1): $x \le y\ \Rightarrow z \to x \le z \to y$;*
(2) *$x \le y\ \Rightarrow z \odot x \le z \odot y$.*

Proof. Let $x \le y$. (1) Since, for all $u \in X$,

$$u \le z \to x\ \Rightarrow\ z \odot u \le x \le y\ \Rightarrow\ z \odot u \le y\ \Rightarrow\ u \le z \to y,$$

we get $z \to x \le z \to y$.

(2) Similarly, for all $u \in X$, we have

$$z \odot y \le u \Rightarrow\ y \le z \to u \Rightarrow\ x \le y \le z \to u \Rightarrow\ x \le z \to u\ \Rightarrow\ z \odot x \le u.$$

This implies $z \odot x \le z \odot y$. □

A half of the next proposition is proved in [6].

Proposition 3.2. *For a w-eo algebra $(X, \to \top)$ with (AP) $(\odot, \to)$, we have $\top \to x = x$ for all $x \in X$ if and only if $\top \odot x = x$ for all $x \in X$.*

Proof. ($\Rightarrow$) Theorem 2.7(2) [6].

($\Leftarrow$) Let $\top \odot x = x$ for all $x \in X$. For any $u \in X$, since

$$u \le \top \to x \Leftrightarrow \top \odot u \le x \Leftrightarrow u \le x \ \ (\forall u \in X),$$

we have $\top \to x = x$ for all $x \in X$. □

It has previously been proven that

Proposition 3.3 (Theorem 3.5 [6])**.** *For any w-eo algebra $(X, \to, \top)$ with (AP) $(\odot, \to)$, it is commutative if and only if the operator $\odot$ is commutative, that is, $x \odot y = y \odot x$ for all $x, y, \in X$.*

Proposition 3.4. *For any w-eo algebra $(X, \to, \top)$, if it is commutative, then it satisfies $\top \to x = x$ for all $x \in X$.*

Proof. Since $\top \to (x \to x) = \top$ and commutativity, we have

$$x \to (\top \to x) = \top.$$

We also have $(\top \to x) \to (\top \to x) = \top$ and thus $\top \to ((\top \to x) \to x) = \top$ and $(\top \to x) \to x = \top$. This means $\top \to x = x$. □

Regarding the condition $\top \to x = x$, we have the following characterization theorem about BCK-algebras.

Theorem 3.5. *Let X be a normal w-eo algebra. X satisfies the condition $\top \to x = x$ for all $x \in X$ if and only if X is a BCK-algebra.*

Proof. ($\Rightarrow$) Let X be a normal w-eo algebra. To show that X is a BCK-algebra, it is sufficient to prove (B2) $x \to ((x \to y) \to y) = \top$, that is, $x \le (x \to y) \to y$. Since X is normal, we have $\top \to x \le (x \to y) \to (\top \to y)$ by (N2). The assumption $\top \to x = x$ for all $x \in X$ implies $x \le (x \to y) \to y$. Thus, X is a BCK-algebra.

($\Leftarrow$) Conversely, we assume that X is a BCK-algebra. It is sufficient to prove $\top \to x = x$ for all $x \in X$. Since X is the BCK-algebra, we get $\top \le (\top \to x) \to x$ and thus $(\top \to x) \to x = \top$. Moreover, since $x \le (x \to x) \to x = \top \to x$ by (B2), we have $x \to (\top \to x) = \top$. Therefore, we obtain $\top \to x = x$ for all $x \in X$. □

For a BCK-algebra X, it is called a *BCK-algebra with condition (S)* [5] if for all $x, y \in X$, there exists a smallest element in $\{t \in X \mid y \le x \to t\}$. Such element is denoted by $x \circ y$. Now, we have the following result which clarifies the relation between eo-algebras with (AP) and BCK-algebras with the condition (S).

Theorem 3.6. *Let X be a normal w-eo algebra satisfying $\top \to x = x$ for all $x \in X$. X satisfies (AP) if and only if X is a BCK-algebra with condition (S).*

Proof. (⇒) Let X be a normal w-eo algebra satisfying $\top \to x = x$ for all $x \in X$. Suppose that X satisfies the condition (AP). Since X is a BCK-algebra, it suffices to show that the condition (S) holds. It follows from (AP) that $y \leq x \to x \odot y$ and $y \leq x \to z \Rightarrow x \odot y \leq z$. This means that a set $\{z \in X \,|\, y \leq x \to z$ has the smallest element $x \odot y$ and hence X satisfies the condition (S).

(⇐) Conversely, assume that X is a BCK-algebra with condition (S). It is sufficient to show that the condition (AP) $x \odot y \leq z \Leftrightarrow y \leq x \to z$ holds for all $x, y, z \in X$. If we define an operator $\odot$ by $x \odot y = x \circ y$ then it is trivial that the operator satisfies (AP). □

A w-eo algebra $(X, \to, \top)$ with (AP) $(\odot, \to)$ is called *associative* [1, 2, 4, 6, 7] if it satisfies the condition

$$x \to (y \to z) = y \odot x \to z \text{ for all } x, y, z \in X.$$

Proposition 3.7. *Let $(X, \to, \top)$ be a w-eo algebra with (AP). Then, it is associative if and only if the operator $\odot$ is associative, that is, $(x \odot y) \odot z = x \odot (y \odot z)$ for all $x, y, z \in X$.*

Proof. (⇒) For all $u \in X$, we have

$$\begin{aligned} (x \odot y) \odot z \leq u \;&\Leftrightarrow\; z \leq (x \odot y) \to u = y \to (x \to u) \\ &\Leftrightarrow\; y \odot z \leq x \to u \\ &\Leftrightarrow\; x \odot (y \odot z) \leq u. \end{aligned}$$

Therefore, $(x \odot y) \odot z = x \odot (y \odot z)$.

(⇐) For all $u \in X$, from

$$\begin{aligned} u \leq x \to (y \to z) \;&\Leftrightarrow\; x \odot u \leq y \to z \\ &\Leftrightarrow\; y \odot (x \odot u) \leq z \\ &\Leftrightarrow\; (y \odot x) \odot u \leq z \\ &\Leftrightarrow\; u \leq (y \odot x) \to z, \end{aligned}$$

we get $x \to (y \to z) = y \odot x \to z$, that is, X is associative.

□

4. Adjoint triple (AT)

In this section we consider properties of w-eo algebras with (AT) $(\to, \odot, \rightsquigarrow)$. We here prove new results which are essential for w-eo algebras with (AT). Let $(X, \to, \top)$ be a w-eo algebra with (AT) $(\to, \odot, \rightsquigarrow)$. Since

$$x \odot y \leq z \;\Leftrightarrow\; y \leq x \to z \;\Leftrightarrow\; x \leq y \rightsquigarrow z \quad (\forall x, y, z \in X),$$

we see that $(\odot, \to)$ is an adjoint pair (AP) from the first equivalence and $(\to, \rightsquigarrow)$ is a Galois pair (GP) from the second one. It also holds the converse. Therefore, a w-eo algebra having (AT) $(\to, \odot, \rightsquigarrow)$ is identical with a w-eo algebra with (AP) $(\to, \odot)$ and (GP) $(\to, \rightsquigarrow)$.

It is easily proved from the definition of (AT).

Proposition 4.1. *Let $(X, \to, \top)$ be a w-eo algebra with (AT) $(\to, \odot, \rightsquigarrow)$. Then it satisfies the following conditions: For all $x, y, z \in X$,*

(1) $x \le y \;\Rightarrow\; z \to x \le z \to y,\ z \rightsquigarrow x \le z \rightsquigarrow y,\ z \odot x \le z \odot y;$

(2) $x \le y \;\Rightarrow\; y \to z \le x \to z,\ y \rightsquigarrow z \le x \rightsquigarrow z,\ x \odot z \le y \odot z.$

By Proposition 3.1, every w-eo algebra with (AP) satisfies (EO1) and every w-eo algebra with (AT) also satisfies (EO2). Consequently, every w-eo algebra with (AT) is an eo-algebra.

Proposition 4.2. *For a w-eo algebra having (AT), it is associative if and only if it satisfies the condition $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$ for all $x, y, z \in X$.*

Proof. $(\Rightarrow)$ Suppose that a w-eo algebra $(X, \to, \top)$ having (AT) is associative. Since, for all $u \in X$,

$$\begin{aligned} u \le x \to (y \rightsquigarrow z) &\Leftrightarrow\; x \odot u \le y \rightsquigarrow z \;\Leftrightarrow\; (x \odot u) \odot y \le z \\ &\Leftrightarrow\; x \odot (u \odot y) \le z \;\Leftrightarrow\; u \odot y \le x \to z \\ &\Leftrightarrow\; u \le y \rightsquigarrow (x \to z), \end{aligned}$$

we get $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.

$(\Leftarrow)$ It is sufficient to show that the operator $\odot$ is associative, that is, $(x \odot y) \odot z = x \odot (y \odot z)$ for all $x, y, z \in X$. For all $u \in X$, it follows from

$$\begin{aligned} (x \odot y) \odot z \le u &\Leftrightarrow\; x \odot y \le z \rightsquigarrow u \\ &\Leftrightarrow\; y \le x \to (z \rightsquigarrow u) = z \rightsquigarrow (x \to u) \\ &\Leftrightarrow\; y \odot z \le x \to u \\ &\Leftrightarrow\; x \odot (y \odot z) \le u \end{aligned}$$

that $(x \odot y) \odot z = x \odot (y \odot z)$. □

Similarly, we have a next result.

Corollary 4.3. *For a w-eo algebra $(X, \to, \top)$ having (AT), it is associative if and only if $x \rightsquigarrow (y \rightsquigarrow z) = x \odot y \rightsquigarrow z$ for all $x, y, z \in X$.*

From the above, we have the following result.

Theorem 4.4. *For a w-eo algebra $(X, \to, \top)$ with (AT), the following conditions are equivalent:*
(1) *X is associative;*
(2) *the operator $\odot$ is associative;*
(3) *$x \rightsquigarrow (y \rightsquigarrow z) = x \odot y \rightsquigarrow z$ for all $x, y, z \in X$;*
(4) *$x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$ for all $x, y, z \in X$;*

Remark 4.5. For a w-eo algebra $(X, \to, \top)$ with (AT) $(\to, \odot, \rightsquigarrow)$, if it is commutative and associative, then $(X, \odot, \top)$ is a commutative monoid with a unit $\top$ and satisfies the residuation

$$x \odot y \leq z \;\Leftrightarrow\; x \leq y \to z \;\Leftrightarrow\; y \leq x \to z.$$

However, we note that the algebra $(X, \odot, \top)$ is not a residuated lattice, because it is not a lattice in general.

A partially ordered set X is called a *join semilattice* if there exists a smallest upper bound of $\{x, y\}$ for all $x, y \in X$. We consider a problem under what condition a w-eo algebra becomes a lattice.

A following formula

$$\text{(sup)} \qquad (x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$$

is called a *sup* condition.

Proposition 4.6. *Let $(X, \to, \top)$ be a w-eo algebra with (AT). If it satisfies the (sup) condition $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$, then*

$$\sup_{\leq}\{x, y\} = (x \to y) \rightsquigarrow y,$$

that is, X is a join semilattice.

Proof. At first we show that $(x \to y) \rightsquigarrow y$ is an upper bound of $\{x, y\}$. From $x \odot (x \to y) \leq y$ we have $x \leq (x \to y) \rightsquigarrow y$. Moreover, since $x \to y \leq \top = y \to y$, we also have $y \odot (x \to y) \leq y$ and thus

$$y \leq (x \to y) \rightsquigarrow y.$$

Therefore, $(x \to y) \rightsquigarrow y$ is the upper bound of $\{x, y\}$.

For any upper bound u of $\{x, y\}$, it follows from $x \leq u$ and Proposition 4.1 that $u \to y \leq x \to y$ and thus

$$(x \to y) \rightsquigarrow y \leq (u \to y) \rightsquigarrow y = (y \to u) \rightsquigarrow u = \top \rightsquigarrow u.$$

From assumption of (AT), X has (GP) and thus $T \rightsquigarrow u = u$ by Proposition 2.3. Therefore, we get $(x \to y) \rightsquigarrow y \leq u$.

From the above, $(x \to y) \rightsquigarrow y$ is the smallest upper bound of the set $\{x, y\}$. ∎

We put $x \vee y = (x \to y) \rightsquigarrow y$. Let $(X, \to, \top)$ be a bounded commutative w-eo algebra with (AT) and (sup) condition, that is, there exists a smallest element $\perp \in X$ with respect to the order $\leq$, or equivalently, it satisfies a condition

$$(\perp) \quad \perp \to x = \top \text{ for all } x \in X.$$

In this case, we note that $x \to \perp = x \rightsquigarrow \perp$. Because, since X has (AT) and is commutative, $x \to \perp = x \rightsquigarrow \perp$ for all $x \in X$ by Proposition 2.6.

Now we define $x' = x \to \perp = x \rightsquigarrow \perp$. It follows from Proposition 4.1 that if $x \leq y$ then $y' \leq x'$ for all $x, y \in X$.

Proposition 4.7. *Let $(X, \to, \top)$ be a bounded commutative w-eo algebra with (AT) and (sup) condition. Then it is a lattice, that is,*

$$\inf_{\leq}\{x, y\} = (x' \vee y')'.$$

Proof. Suppose that $(X, \to, \top)$ is a bounded commutative w-eo algebra with (AT) and (sup) condition. For all $x \in X$, we have $x'' = x$ because

$$x'' = (x \to \perp) \rightsquigarrow \perp = (\perp \to x) \rightsquigarrow x = \top \rightsquigarrow x = \top \to x = x$$

by Proposition 3.4.

Since $x', y' \leq x' \vee y'$ and $(x' \vee y')' \leq x'' = x, y'' = y$, that is, $(x' \vee y')'$ is a lower bound of $\{x, y\}$. For any lower bound z of $\{x, y\}$, since $z \leq x, y$, we have $x', y' \leq z'$ and $x' \vee y' \leq z'$. This implies $z = z'' \leq (x' \vee y')'$. Therefore, $\inf_{\leq}\{x, y\} = (x' \vee y')'$.

□

We put $x \wedge y = (x' \vee y')'$. It follows that every bounded commutative w-eo algebra with (AT), (sup) and $(\perp)$ is a lattice and hence that it becomes an involutive residuated lattice.

A w-eo algebra $(X, \to, \top)$ is called *idempotent* if it satisfies the condition

$$x \to (x \to y) = \top \;\Leftrightarrow\; x \to y = \top.$$

Proposition 4.8. *Let $(X, \to, \top)$ be an associative w-eo algebra with (AT). Then it is idempotent if and only if the operator $\odot$ is idempotent, that is, $x \odot x = x$ for all $x \in X$.*

Proof. $(\Rightarrow)$ Since, for all $u \in X$,

$$x \odot x \leq u \Leftrightarrow\; x \leq x \to u \Leftrightarrow\; x \to (x \to u) = \top \Leftrightarrow\; x \to u = \top \Leftrightarrow\; x \leq u,$$

we get $x \odot x = x$.

(⇒) Conversely, suppose $x \odot x = x$. Then we have

$$x \to (x \to y) = \top \;\Leftrightarrow\; x \odot x \to y = \top \;\Leftrightarrow\; x \to y = \top.$$

This says that X is idempotent. □

From these results, we have a characterization theorem of Boolean algebras by w-eo algebras.

Theorem 4.9. *Let $(X, \to, \top)$ be a w-eo algebra. Then, $(X, \to, \top)$ is bounded, commutative, associative, idempotent with (sup) condition and has (AT) if and only if it is a Boolean algebra.*

Proof. (⇒) Let $(X, \to, \top)$ be a w-eo algebra satisfying the conditions in the theorem. In this case we note that $(X, \wedge, \vee, \to, \odot, \bot, \top)$ is an involutive (commutative) residuated lattice. We also note that an involutive residuated lattice becomes a Boolean algebra if $x \wedge y = x \odot y$ for all $x, y \in X$, because if $x \wedge y = x \odot y$ holds for all $x, y \in X$ in any residuated lattice, then X is a Heyting algebra and satisfies $x'' = x$ for all $x \in X$ ([3, 8]). Therefore $(X, \wedge, \vee, \to, \odot, \bot, \top)$ is the Boolean algebra.

For all $x, y \in X$, we have $x \odot y \leq x$ by $y \leq \top = x \to x$. Moreover, since $\odot$ is commutative, we get $x \odot y \leq y$ and hence $x \odot y \leq x \wedge y$. On the other hand, we have

$$x \wedge y = (x \wedge y) \odot (x \wedge y) \leq x \odot y.$$

Therefore, $x \wedge y = x \odot y$ for all $x, y \in X$.

(⇐) It is trivial. □

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References

1. M. E. Della Stella and C. Guido, Associativity, Commutativity and symmetry in residuated structures, *Order*, **30** (2013), 363–401.
2. M. E. Della Stella and C. Guido, Extended-order algebras and fuzzy implicators, *Soft Computing*, **16** (2012), 1883–1892.
3. N. Galatos, P. Jipsen, T. Kowalski and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Elsevier, 2007.
4. C. Guido and P. Toto, Extended-order algebras, *Journal of Applied Logic*, **6** (2008), 609–626.

5. K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, *Math. Japonica*, **23** (1978), 1–26.
6. Y. C. Kim and J. M. Ko, Some properties of extended order algebras, *Korean J. Math.*, **21** (2013), 63–73.
7. Y. C. Kim and J. M. Ko, Weak extended order algebras having adjoint triples, *Korean J. Math.*, **21** (2013), 351–363.
8. H. Rasiowa, *An algebraic approach to non-classical logics*, Studies in Logic and the Foundations of Mathematics, **78**, Amsterdam, 1974.
9. E. Sameri and R. A. Borzooei, Extended implicative groupoids, *Journal of intelligent and fuzzy systems*, **40** (2021), 1261–1275.
10. E. Sameri, R. A. Borzooei and D. E. Bagha, Filter theory of extended implicative groupoids, *Soft Computing* **25** (2021), 14499–14508.
11. A. Wroński, BCK-algebras do not form a variety, *Math. Japonica*, **28** (1983), 211–213.

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ON WEAK EXTENDED ORDER ALGEBRAS WITH ADJOINT PAIRS AND GALOIS PAIRS

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بررسی جبرهای مرتب توسعه‌یافته‌ی ضعیف با جفت‌های الحاقی و جفت‌های گالوا

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در این مقاله، ویژگی‌های جبرهای مرتب توسعه‌یافته‌ی ضعیف با جفت‌های الحاقی و جفت‌های گالوا را بررسی می‌کنیم و نتایج جدیدی را اثبات می‌کنیم. به‌علاوه، رابطه‌ی بین این جبرها و BCK-جبرها را مشخص می‌کنیم، یعنی؛ در واقع نشان می‌دهیم کلاس همه‌ی جبرهای مرتب توسعه‌یافته‌ی ضعیف نرمال با جفت الحاقی که در شرط $\top \to x = x$ صدق می‌کنند و کلاس همه‌ی BCK-جبرها با شرط (S) یکسان هستند.

کلمات کلیدی: جبرهای مرتب توسعه‌یافته (ضعیف)، سه‌گانه‌های الحاقی، جفت‌های الحاقی، جفت‌های گالوا، BCK-جبرها.