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ON THE DIAMETER AND CONNECTIVITY OF BIPARTITE KNESER TYPE-k GRAPHS

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Abstract. Let $n \in \mathbb{Z}^+$, $n > 1$, and k be an integer, $1 \leq k \leq n-1$. The graph $H_T(n,k)$ is defined as a graph with vertex set V, as all non-empty subsets of $\mathcal{S}_n = \{1,2,3,\ldots,n\}$. It is a bipartite graph with partition (V_1, V_2), in which V_1 contains the k-element subsets of $\mathcal{S}_n$ and V_2 contains i-element subsets of $\mathcal{S}_n$, for $1 \leq i \leq n,\ i \neq k$. An edge exists between a vertex $U \in V_1$ and a vertex $W \in V_2$ if either $U \subset W$ or $W \subset U$. In this paper, we established formulae for the number of edges, vertex connectivity, edge connectivity and degree polynomial. Then, we analysed the clique number and diameter of $H_T(n,k)$. We also verified that the graphs $H_T(n,k)$ and $H_T(n,n-k)$ are isomorphic. Degree-based topological indices such as the general Randic connectivity index, the first general Zagreb index and the general sum connectivity index are also computed. Also, we proved that the line graph of $H_T(n,k)$ is not a bipartite graph.

1. Introduction

For a graph $G = (V, E)$ with vertex set V and edge set E, the degree of a vertex v denoted by d_v, is the number of edges incident on it. The minimum and maximum degree of a graph G are $\delta(G) = \min_{v \in V(G)} \{d_v\}$ and $\Delta(G) = \max_{v \in V(G)} \{d_v\}$, respectively. The concept of polynomial representation of a graph was introduced in [4]. If a_i is the number of vertices having degree i, then the degree polynomial representation of a graph G is defined as $f_G(x) = \sum_{i=1}^{\Delta(G)} a_i x^i$.

Let n and k be fixed positive integers with $n \geq k$. The Kneser graph $K(n,k)$ is defined as a graph with k-subsets of a fixed set with n elements as vertices, and the adjacency exists between two disjoint k-subsets [8]. The diameter of the Kneser graph $K(2n+k, n)$ was studied by Valencia Pabon and Juan Carlos vera [17], and is given by $\lceil \frac{n-1}{k} \rceil + 1$. The graph $K(5,2)$ is isomorphic to the Petersen graph. On the basis of the Kneser graph, a

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generalised Kneser graph $K(n,r,s)$ for $n = 2r - s + k, where\ r, s < r, k \in \mathbb{Z}^+$, is defined as a graph with the vertex set as r-subsets of $\{1, 2, \ldots, n\}$ and an edge between two vertices U and W exists if $|U \cap W| \leq s$ [11]. If $s = 0$, then $K(n,r,s)$ is the Kneser graph $K(n,r)$. In [6], the diameter of $K(n,r,s)$ is given by $\lceil \frac{r-s-1}{s+k} \rceil + 1$.

The bipartite Kneser graph, $H(n,k)$ [12] for a positive integer $n > 1$ has a vertex set consisting of all k-element and $n-k$ element subsets of $\mathcal{S}_n = \{1, 2, 3, \ldots, n\}$, and an edge between any two vertices U and W exists when $U \subset W\ or\ W \subset U$. When $n = 2k$, $H(n,k)$ is a null graph and so it was assumed that $n \geq 2k+1$. The bipartite Kneser graphs and their properties have wide applications in coding theory [7], [2]. The sufficient conditions for the equality of edge-connectivity and minimum degree of a bipartite graph were proved in [13].

In [16], bipartite Kneser type-1 graph, $H_T(n,1)$ was introduced and some algebraic properties were investigated. In connection with this, bipartite Kneser type-k graph $H_T(n,k)$ [16] is defined as a graph with partition(V_1, V_2), in which V_1 contains k-element subsets of $\mathcal{S}_n, n > 1$, and V_2 contains all other non-empty subsets of $\mathcal{S}_n$. The edge set is defined as

$$\big\{UW : U \in V_1, W \in V_2, \text{ and } UW \text{ exists if and only if } U \subset W\ or\ W \subset U\big\}.$$

It is proved that the bipartite Kneser type-k graphs have $n!$ automorphisms and that the automorphism group is isomorphic to the symmetric group S_n for each $n \geq 3$, $2k \neq n$. If n is even, $n = 2k$, then the automorphism group of $H_T(n,k)$ is isomorphic to $S_n \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is a cyclic group of order 2. The authors also investigated the diameter of $H_T(n,1)$ as 4, for $n \geq 4$. A possible $2S3$ transformation of a similar graph and its properties were studied in [15].

There are various types of topological indices such as degree based indices, distance based indices and spectrum based indices. A study on determining the distance spectrum of a class of distance integral graphs was done by Mirafzal and Kogani [10]. Degree based indices [1] are very important in chemical graph theory, and in [14], the authors mentioned certain degree-based topological indices of Boron B_{12}. In [3], authors introduced the entire Wiener index of a graph and obtained exact values of this index for trees and some graph families. Also, they established some properties and bounds for the entire Wiener index.

In this paper, we generalized the diameter of $H_T(n,k)$ for $n > 3$ and $1 \leq k \leq n-1$. Also, we established general formulae for finding the number of edges, degree sequence, degree polynomial representation, vertex connectivity and edge connectivity. We identified the clique number as 2 for the graph $H_T(n,k)$. We proved that the graphs $H_T(n,k)$ and $H_T(n,n-k)$ are isomorphic. Degree-based topological indices such as the general Randic connectivity index, the first general Zagreb index and general sum connectivity index are also computed.

2. Preliminaries

For a connected graph G that is not complete, a vertex-cut is a set S of vertices of G such that $G-S$ is disconnected. The cardinality of a minimum vertex-cut is the vertex connectivity $\kappa(G)$ of G. The cardinality of the minimum number of edges whose removal disconnect the graph is the edge connectivity, $\lambda(G)$ [5]. The distance $d(u,v)$ between any two distinct vertices u and v in a connected graph G, is the length of the shortest path between them, and the diameter is given by $diam(G) = \max_{u,v\in V(G)}\{d(u,v)\}$. If every pair of vertices in a graph G is adjacent, then G is complete, and the order of the largest complete subgraph is the clique number, $\omega(G)$ [5].

Lemma 2.1. [9], [14] *For a simple graph G, let d_u denote the degree of the vertex u in $V(G)$ and α be any real number. Then*

1. *The general Randic index,* $R_\alpha(G) = \sum_{uv\in E(G)} \big(d_u d_v\big)^\alpha$.
2. *The first general Zagreb index,* $M_\alpha(G) = \sum_{u\in V(G)} (d_u)^\alpha$.
3. *The general sum connectivity index,* $\chi_\alpha(G) = \sum_{uv\in E(G)} (d_u+d_v)^\alpha$.

Theorem 2.2. [5] *A non-trivial graph G is a bipartite graph if and only if G contains no odd cycles.*

Let $\mathcal{S}_n = \{1,2,3,\dots,n\}$ and $1\le k\le n-1$. The bipartite Kneser type-k graph $H_T(n,k)$ is obtained by partitioning the set of all non-empty subsets V of $\mathcal{S}_n$ into two partite sets, V_1 containing k-element subsets and $V_2 = V - V_1$. An edge between two vertices $U\in V_1$ and $W\in V_2$ exists if either $U\subset W$ or $W\subset U$. The graph $H_T(n,k)$ is connected but not regular.
Throughout this paper, n is a positive integer greater than 1 and $1\le k\le n-1$. The partite set V_1 consists of all k-element subsets of $\mathcal{S}_n$ and V_2 consists of all other non-empty i-element subsets of $\mathcal{S}_n$, $i\ne k$. The number of k-element subsets of $\mathcal{S}_n$ is $\binom{n}{k}$.

Theorem 2.3. [16] *The diameter of the graph $H_T(n,1)$ is 4 for $n\ge 4$. .*

Proposition 2.4. [16] *For the graph $H_T(n,1)$, let $d_h(u,v)$ denote the number of unordered pairs of vertices for which $d(u,v)=h$. Then*

$$d_h(u,v)=\begin{cases} n\cdot 2^{n-1}-n & \text{if } h=1,\\ \binom{n}{2}+\left[\binom{2^n-n-1}{2}-\frac{1}{2}\sum_{r=2}^{n-2}\left[\binom{n}{r}\sum_{k=2}^{n-2}\binom{n-r}{k}\right]\right] & \text{if } h=2,\\ n(2^{n-1}-n) & \text{if } h=3,\\ \frac{1}{2}\sum_{r=2}^{n-2}\left[\binom{n}{r}\sum_{k=2}^{n-2}\binom{n-r}{k}\right] & \text{if } h=4. \end{cases}$$

3. Main results

Connectivity and diameter of bipartite Kneser type-k graphs

The bipartite Kneser type-k graph, $H_T(n,k)$ has 2^n-1 vertices as we consider only non-empty subsets of $\mathcal{S}_n$. To find the size of $H_T(n,k)$, we can use the following proposition.

Proposition 3.1. *Number of edges of the graph $H_T(n,k)$ is $(2^k+2^{n-k}-3)\binom{n}{k}$.*

Proof. From the definition of $H_T(n,k)$, the partite set V_1 has $\binom{n}{k}$ elements, say U_j for $j=1,2,3,\ldots,\binom{n}{k}$. Each U_j has 2^k-2 incident edges from vertices $W_i,\ i<k$ in V_2. For $i>k$, the subsets containing each U_j are also adjacent to it, and there are $\binom{n-k}{1}+\binom{n-k}{2}+\cdots+\binom{n-k}{n-k}=2^{n-k}-1$ such edges. Thus, the number of edges incident on each U_j is $2^k-2+2^{n-k}-1=2^k+2^{n-k}-3$. Hence, the total number of edges in $H_T(n,k)$ is $(2^k+2^{n-k}-3)\binom{n}{k}$. □

Since the degree of a vertex is the number of edges incident on it, $\Delta(H_T(n,k)) = \max\{2^k+2^{n-k}-3, \binom{n}{k}\}$. Hence, the degree sequence is obtained as follows.

Lemma 3.2. *For the graph $H_T(n,k)$,*

(1) *For each $i, 1\le i\le k-1$, $\binom{n}{i}$ vertices have degrees $\binom{n-i}{k-i}$.*
(2) *$\binom{n}{k}$ vertices have $2^k+2^{n-k}-3$ degrees.*
(3) *For each $i,\ k+1\le i\le n-1$, $\binom{n}{i}$ vertices have degrees $\binom{i}{k}$.*

Proof. As the same arguments given in Proposition 3.1, for vertices in V_1, $\binom{n}{k}$ vertices have degrees $2^k+2^{n-k}-3$. By considering vertices in V_2, if $i<k$, $\binom{n}{i}$ vertices have degrees $\binom{n-i}{k-i}$; and if $i>k$, $\binom{n}{i}$ vertices have degrees $\binom{i}{k}$. □

The degree polynomial representation of the graph $H_T(n,k)$ is given by the following theorem.

Theorem 3.3. *The degree polynomial of the graph $H_T(n,k)$,*

$$P(H_T(n,k)) = \sum_{i=1}^{k-1} \binom{n}{i} x^{\binom{n-i}{k-i}} + \binom{n}{k} x^{2^k+2^{n-k}-3} + \sum_{i=k+1}^{n} \binom{n}{i} x^{\binom{i}{k}}.$$

Proof. The proof follows from Lemma 3.2 and definition of the graph $H_T(n,k)$. □

The value of the polynomial when $x = 1$ represents the number of vertices and that of the derivative of $P(H_T(n,k))$ represents twice the number of edges in $H_T(n,k)$.

It is observed that the number of vertices, the number of edges, degree sequence, degree polynomial representation, the number of vertex pairs with certain distance are all the same for the graphs $H_T(n,k)$ and $H_T(n,n-k)$. So we have the following theorem.

Theorem 3.4. *The bipartite Kneser type-k graphs, $H_T(n,k)$ and $H_T(n,n-k)$ are isomorphic.*

Proof. Define a map ϕ from $V(H_T(n,k))$ to $V(H_T(n,n-k))$ by

$$\phi(U_i) = U_i',\ U_i \in V_1\ of\ V(H_T(n,k)), 1 \leq i \leq \binom{n}{k},$$

$$U_i' \in V_1\ of\ V(H_T(n,n-k)), 1 \leq i \leq \binom{n}{n-k}$$

$$\phi(W_j) = \begin{cases} W_{n-j} & if\ W_j \in V_2,\ j \neq n, k, \\ W_j & if\ j = n. \end{cases}$$

It is clear that ϕ is one to one, onto and preserves adjacency.
Hence, $H_T(n,k) \cong H_T(n,n-k)$ □

We obtained the following results for the vertex connectivity and edge connectivity for the graph $H_T(n,k)$.

Theorem 3.5. *For any integer $n > 2$ and $1 \leq k \leq n-1$, vertex connectivity,*

$$\kappa(H_T(n,k)) = \begin{cases} k+1 & for\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\ \kappa(H_T(n,n-k)) & for\ \lfloor \frac{n}{2} \rfloor < k \leq n-1. \end{cases}$$

Proof. Vertex connectivity of a graph G is the minimum cardinality of the set of vertices U of G such that $G - U$ is disconnected. Also, $0 \leq \kappa(G) \leq n-1$ and $\kappa(G) \leq \delta(G)$. Since $H_T(n,k)$ is connected and $n > 2$, $\kappa(G)$ is neither 0 nor 1.

As $H_T(n,k)$ is a bipartite graph, it is enough to find out the vertices with minimum degree so that the removal of the vertices adjacent to the vertex with minimum degree will disconnect the graph. From Lemma 3.2, if $i < k$, there are $\binom{n-i}{k-i}$ elements in V_1 adjacent to any $\binom{n}{i}$ vertices in V_2.

If $i > k$, there exist $\binom{i}{k}$ vertices in V_1 adjacent to any $\binom{n}{i}$ vertices in V_2 and its removal from V_1 separates the corresponding vertex in V_2 and hence the graph is disconnected. Then $min\{\binom{n-i}{k-i}_{i<k}, \binom{i}{k}_{i>k}\}$ will be the vertex connectivity.

Case 1: Let $1 \leq k \leq \lfloor\frac{n}{2}\rfloor$. Since $\binom{n}{k+1}$ in V_2 has exactly $\binom{k+1}{k} = k+1$ adjacent vertices in V_1, the removal of these $k+1$ vertices from V_1 disconnect the graph.

Case 2: Let $\lfloor\frac{n}{2}\rfloor < k \leq n-1$. In this case,

$$\begin{aligned} min\left\{\binom{n-i}{k-i}_{i<k}, \binom{i}{k}_{i>k}\right\} &= \binom{n-(k-1)}{1} \\ &= n-(k-1) \\ &= (n-k)+1 \\ &= \kappa(H_T(n,n-k)) \end{aligned}$$

. □

Corollary 3.6. *For any integer $n > 2$ and $1 \leq k \leq n-1$, the edge connectivity*

$$\lambda(H_T(n,k)) = \begin{cases} k+1 & \text{for } 1 \leq k \leq \lfloor\frac{n}{2}\rfloor, \\ \lambda(H_T(n,n-k)) & \text{for } \lfloor\frac{n}{2}\rfloor < k \leq n-1. \end{cases}$$

Proof. As the graph is connected and $n > 2$, the edge connectivity cannot be 0 and 1. Also, we know that for any connected graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

For the graph $H_T(n,k)$, degree of vertices is obtained from Lemma 3.2. From the above theorem 3.5, the removal of the edges incident on the vertex with minimum degree must disconnect the graph.

Case 1: Let $1 \leq k \leq \lfloor\frac{n}{2}\rfloor$. The $\binom{n}{k+1}$ vertices in V_2 have minimum degree with $k+1$ incident edges. If we remove these $k+1$ edges, the graph becomes disconnected into two components, one containing $\binom{n}{k+1}$ vertices and the other containing $H_T(n,k) - \binom{n}{k+1}$. Thus, the edge connectivity is $k+1$.

Case 2: Let $\lfloor\frac{n}{2}\rfloor < k \leq n-1$. Here, the number of edges incident on any $k-1$ element subset of $\mathcal{S}_n$ is

$\binom{n-(k-1)}{1} = n-(k-1) = (n-k)+1 = \lambda(H_T(n,n-k))$.

□

Corollary 3.7. *The vertex connectivity for the graph $H_T(2,1)$ is 1.*

Corollary 3.8. *The edge connectivity for the graph $H_T(2,1)$ is 1.*

Observation 1. The graph $H_T(n,k)$ is a connected, bipartite graph for any n and $1 \leq k \leq n-1$. So, the clique number of $H_T(n,k) = 2$.

Theorem 3.9. *The diameter of the graph $H_T(n,k)$, for $n > 3$ and $1 \leq k \leq n-1$ is 4.*

Proof. Diameter of a connected graph G is the greatest distance between any two vertices of G. The graph $H_T(n,k)$ has $2^n - 1$ vertices.

$d(U_i, W_j) = 1$ if $U_i \in V_1$ for $1 \leq i \leq \binom{n}{k}$ and $W_j \in V_2$ for $1 \leq j \leq n-1, j \neq k$ are adjacent.

Suppose there exist non adjacent vertices say $U_1 \in V_1$ and $W_1 \in V_2$. Since $\mathcal{S}_n$ is a vertex in V_2 and $U_1 \subset \mathcal{S}_n$, there is an edge $U_1\mathcal{S}_n$. Also, V_1 has a $k-$element subset U_i , $1 \leq i \leq \binom{n}{k}$ of $\mathcal{S}_n$ such that either $W_1 \supset U_i$ or $U_i \supset W_1$. Then $U_1\mathcal{S}_nU_iW_1$ will be the shortest path of length 3 connecting the vertices U_1 and W_1. Thus, $d(U_1, W_1) = 3$ if $U_1 \in V_1$ and $W_1 \in V_2$ are non-adjacent.

The presence of $\mathcal{S}_n$ as a vertex in V_2 always yields a path $U_i\mathcal{S}_nU_j$ for vertices U_i and $U_j \in V_1$, $1 \leq i,j \leq \binom{n}{k}$. In such cases, $d(U_i, U_j) = 2$.

Now, consider the vertex pair (W_i, W_j) in V_2 for $1 \leq i,j \leq n, i \neq k$ and $j \neq k$ such that they have no common $k-$element subsets in V_1. Let U_k adjacent to W_i, and $U_{k'}$ adjacent to W_j where $1 \leq k,k' \leq \binom{n}{k}$. The path $W_iU_k\mathcal{S}_nU_{k'}W_j$ will connect the vertices W_i and W_j. Thus, $d(W_i, W_j) = 4$. Hence, $diam(H_T(n,k)) = 4$. □

Corollary 3.10. *Diameter of $H_T(3,k) = 3$ for $k = 1, 2$.*

In order to find out certain distance based topological indices, we need the following results.

Lemma 3.11. *For the graph $H_T(n,k)$, the number of unordered pairs of vertices with distance h, $d_h(U,W)$ is as follows:*

$$d_h(U,W) = \begin{cases} (2^k + 2^{n-k} - 3)\binom{n}{k} & \text{if } h = 1, \\ \binom{n}{k}(2^n - 1 - \binom{n}{k}) - (2^k + 2^{n-k} - 3)\binom{n}{k} & \text{if } h = 3. \end{cases}$$

Proof. The distance between any pair (U, W) of vertices, $U \in V_1$ and $W \in V_2$ is either 1 or 3. There are $\binom{n}{k}(2^n - 1 - \binom{n}{k})$ vertex pairs having distances 1 and 3. If there is an edge between the vertices, then $d_h(U, W) = 1$ and there are $(2^k + 2^{n-k} - 3)\binom{n}{k}$ such vertex pairs. Thus, the number of vertices with distance 3 is $\binom{n}{k}(2^n - 1 - \binom{n}{k}) - (2^k + 2^{n-k} - 3)\binom{n}{k}$. □

Lemma 3.12. *For the graph $H_T(n, k)$, the number of unordered pairs of vertices with distances 2 and 4 is given by the equation*

$$d_2(U, W) + d_4(U, W) = \binom{\binom{n}{k}}{2} + \binom{(2^n - 1 - \binom{n}{k})}{2}.$$

Proof. Let (V_1, V_2) be the partition of the graph $H_T(n, k)$ as given in the definition. So, V_1 contains $\binom{n}{k}$ vertices. Since $\mathcal{S}_n$ itself is an element in V_2, there exists a $U_i\mathcal{S}_nU_j$ path between any pair of vertices (U_i, U_j) for $U_i, U_j \in V_1$. Also, the distance between any pair (W_i, W_j) for $W_i, W_j \in V_2$ is either 2 or 4. The part V_2 has $2^n - 1 - \binom{n}{k}$ vertices. Hence, the total number of vertex pairs with distance $h = 2$ and $h = 4$ is $\binom{\binom{n}{k}}{2} + \binom{2^n-1-\binom{n}{k}}{2}$. □

Lemma 3.13. *For the graph $H_T(n, 2)$, the number of unordered pairs of vertices with distance 2 is given by the sum*

$$D(n,2) = \begin{cases} \displaystyle\sum_{j=1,2} \binom{\binom{n}{j}}{2} + \sum_{j=3}^{n} j\binom{n}{j} + \frac{1}{2}\sum_{j=3}^{n-1}\binom{n}{j}\left[\sum_{r=2}^{j-1}\binom{j}{r}\binom{n-j}{j-r}\right] + \\ \displaystyle\sum_{i=3}^{n-2}\left\{\sum_{j=i+1}^{n-1}\binom{n}{j}\left[\sum_{p=2}^{i}\binom{j}{p}\binom{n-j}{i-p}\right]\right\} + \sum_{i=3}^{n-1}\binom{n}{i}. \end{cases}$$

Proof. Consider the bipartite Kneser type-2 graph $H_T(n, 2)$. The vertex partition V_1 contains only 2−element subsets of $\mathcal{S}_n$, and V_2 contains both non-empty subsets and supersets of elements in V_1. Let $|U|$ denote the cardinality of the vertex U. For any positive integer n, the vertex pairs (U, W) with distance 2 and the number of such pairs, say, l_ι can be found in the following way.

- If both U and $W \in V_1$, then there exists a path $U\mathcal{S}_nW$ such that $d(U, W) = 2$ and if $U, W \in V_2$ and $|U| = |W| = 1$, then $d(U, W) = 2$.

  Number of such pairs of vertices is $l_1 = \displaystyle\sum_{j=1,2}\binom{\binom{n}{j}}{2}$.

  In the remaining cases, the vertex pair $(U, W) \in V_2$.

- If $|U| = 1$ and $|W| = j$, where $3 \leq j \leq n$, then

$$l_2 = \sum_{j=3}^{n} \binom{n}{j} \binom{j}{1}$$

.

- $|U| = |W| = j,\ 3 \leq j \leq n-1$, then

$$l_3 = \frac{1}{2} \sum_{j=3}^{n-1} \binom{n}{j} \left[\sum_{r=2}^{j-1} \binom{j}{r} \binom{n-j}{j-r} \right]$$

.

- If $i = |U| < |W| = j$, where $3 \leq i \leq n-2$ and $i+1 \leq j \leq n-1$,

$$l_4 = \begin{cases} \sum_{i=3}^{n-2} \left\{ \sum_{j=i+1}^{n-1} \binom{n}{j} \left[\sum_{p=2}^{i} \binom{j}{p} \binom{n-j}{i-p} \right] \right\} & for\ n > 4, \\ \\ 0 & otherwise. \end{cases}$$

- If $3 \leq |U| \leq n-1$ and $W = \mathcal{S}_n$, then $l_5 = \sum_{i=3}^{n-1} \binom{n}{i}$

Then, $D(n,2) = d_h(U,W) = l_1 + l_2 + l_3 + l_4 + l_5$ when $h = 2$. □

From the Theorem 3.9, it is clear that the maximum distance in the graph $H_T(n,k)$, $1 \leq k \leq n-1$, $n > 3$, is 4. Therefore, we get the following result.

Theorem 3.14. *For the graph $H_T(n,2)$, the number of unordered pairs of vertices with distance h, $d_h(U,W)$ is as follows:*

$$d_h(U,W) = \begin{cases} (2^{n-2}+1)\binom{n}{2} & \text{if } h = 1, \\ D(n,2) & \text{if } h = 2, \\ \binom{n}{2}\left[3 \times 2^{n-2} - \binom{n}{2} - 2\right] & \text{if } h = 3, \\ \binom{\binom{n}{2}}{2} + \binom{(2^n-1-\binom{n}{2})}{2} - D(n,2) & \text{if } h = 4, \end{cases}$$

where $D(n,2)$ is as given in the Lemma 3.13.

Proof. The proof follows from the Lemmas 3.11, 3.12 and 3.13. □

Lemma 3.15. *For the graph $H_T(n,3)$, the number of unordered pairs of vertices with distance 2 is given by the sum*

$$D(n,3) = \begin{cases} \sum_{j=1,k} \binom{\binom{n}{j}}{2} + \sum_{i=1}^{k-1} \sum_{j=k+1}^{n} \binom{n}{j}\binom{j}{i} + \\ \frac{1}{2} \sum_{j=k+1}^{n-1} \binom{n}{j} \left[\sum_{r=k}^{j-1} \binom{j}{r}\binom{n-j}{j-r} \right] + \\ \sum_{i=k+1}^{n-2} \left[\sum_{j=i+1}^{n-1} \binom{n}{j} \left[\sum_{p=k}^{i} \binom{j}{p}\binom{n-j}{i-p} \right] \right] + \\ \sum_{i=k+1}^{n-1} \binom{n}{i} + n(n-1)^2 \end{cases}$$

Proof. Consider the bipartite Kneser type-k graph $H_T(n,k)$ when $k=3$; In addition to the cases in the Lemma 3.13, there is a situation such that $U, W \in V_2$ with

- $|U| = 1$ and $|W| = 2$
- $|U| = |W| = 2$
- $|U| = 2,\ 4 \leq |W| \leq n$

Thus, for the first two cases we have $l'_2 = \binom{n}{2}\binom{2}{1}\left[1 + \binom{n-2}{1}\right] = n(n-1)^2$ number of such vertex pairs and for the third case, it will be $\sum_{i=1}^{k-1} \sum_{j=k+1}^{n} \binom{n}{j}\binom{j}{i}$.

Thus, for $k=3$, the total number of vertex pairs with distance 2 is

$$D(n,3) = \begin{cases} \sum_{j=1,k} \binom{\binom{n}{j}}{2} + \sum_{i=1}^{k-1} \sum_{j=k+1}^{n} \binom{n}{j}\binom{j}{i} + \\ \frac{1}{2} \sum_{j=k+1}^{n-1} \binom{n}{j} \left[\sum_{r=k}^{j-1} \binom{j}{r}\binom{n-j}{j-r} \right] + \\ \sum_{i=k+1}^{n-2} \left[\sum_{j=i+1}^{n-1} \binom{n}{j} \left[\sum_{p=k}^{i} \binom{j}{p}\binom{n-j}{i-p} \right] \right] + \\ \sum_{i=k+1}^{n-1} \binom{n}{i} + n(n-1)^2 \end{cases}$$

□

Thus, by Lemmas 3.11, 3.12 and 3.15, we have the following theorem.

Theorem 3.16. *For the graph $H_T(n,k)$, $k=3$, the number of unordered pairs of vertices with distance h, $d_h(U,W)$ is as follows:*

$$d_h(U,W)=\begin{cases}(5+2^{n-3})\binom{n}{3} & \text{if } h=1,\\ D(n,3) & \text{if } h=2,\\ \binom{n}{3}\left[7\times 2^{n-3}-\binom{n}{3}-6\right] & \text{if } h=3,\\ \binom{\binom{n}{3}}{2}+\binom{2^n-1-\binom{n}{3}}{2}-D(n,3) & \text{if } h=4,\end{cases}$$

where $D(n,3)$ is as in the Lemma 3.15.

Proof. For the graph $H_T(n,3)$, by substituting $k=3$ in the Lemma 3.11 and using the Lemma 3.15, we get the result. □

The line graph of the graph $H_T(n,k)$ denoted by $L(n,k)$ is the graph with vertex set

$$V=\left\{(U_iW_j):U_i\in V_1(H_T(n,k)),1\le i\le\binom{n}{k};W_j\in V_2(H_T(n,k)),\right.$$
$$\left.1\le j\le n-1,j\ne k \text{ and } U_i\subset W_j \text{ or } W_j\subset U_i\right\}.$$

Two vertices in $L(n,k)$ are adjacent if and only if the corresponding edges of $H_T(n,k)$ are adjacent in $H_T(n,k)$.

We studied the properties of $L(n,k)$ and obtained the result:

Theorem 3.17. *The line graph $L(n,k)$ of the graph $H_T(n,k)$ is not a bipartite graph.*

Proof. Let U_iW_i denote an edge in the graph $H_T(n,k)$. Corresponding to each edge $U_i\mathcal{S}_n$ in the graph $H_T(n,k)$, the line graph $L(n,k)$ must have vertices $(U_i\mathcal{S}_n)$ for $1\le i\le\binom{n}{k}$. Then, $L(n,k)$ must contain a cycle $(U_i\mathcal{S}_n),(U_{i+1}\mathcal{S}_n)$, $(U_{i+2}\mathcal{S}_n),(U_i\mathcal{S}_n)$ of length 3, starting from any i for $1\le i\le\binom{n}{k}$. Thus, using the Theorem 2.2, $L(n,k)$ is not a bipartite graph. □

4. Some degree-based topological indices of $H_T(n,k)$

In chemical graph theory, certain topological indices are used to study certain physico-chemical properties of molecules so that it converts molecular graph into a real number. One of the main topological indices is degree-based topological indices. Here, we discussed some degree-based topological indices of the graph $H_T(n,k)$.

Theorem 4.1. *Let α be any real number. Then, for the graph $H_T(n,k)$, for any positive integer n and $1 \leq k \leq n-1$,*

(1) *The general Randic connectivity index is given by*

$$R_\alpha(H_T(n,k)) = \begin{cases} \left(2^k + 2^{n-k} - 3\right)^\alpha \left[\sum_{i=1}^{k-1} \binom{n}{i}\binom{n-i}{k-i}\left(\binom{n-i}{k-i}\right)^\alpha \right. \\ \left. + \sum_{i=k+1}^{n} \binom{n}{i}\binom{i}{k}\left(\binom{i}{k}\right)^\alpha \right] \end{cases}$$

(2) *The general sum connectivity index,*

$$\chi_\alpha(H_T(n,k)) = \begin{cases} \sum_{i=1}^{k-1} \binom{n}{i}\binom{n-i}{k-i}\left[\left(2^k + 2^{n-k} - 3\right) + \binom{n-i}{k-i}\right]^\alpha \\ + \sum_{i=k+1}^{n} \binom{n}{i}\binom{i}{k}\left[\left(2^k + 2^{n-k} - 3\right) + \binom{i}{k}\right]^\alpha \end{cases}$$

Proof. The graph $H_T(n,k)$ has $2^n - 1$ vertices and $(2^k + 2^{n-k} - 3)\binom{n}{k}$ edges. Using Lemma 3.2, the edge set $E(H_T(n,k))$ can be expressed as the union of disjoint edge sets

$$E_i = \left\{UW : d_U = 2^k + 2^{n-k} - 3,\ d_W = \binom{n-i}{k-i},\ 1 \leq i \leq k-1 \right.$$
$$\left. \text{or } d_U = 2^k + 2^{n-k} - 3,\ d_W = \binom{i}{k},\ k+1 \leq i \leq n\right\}.$$

There are $\binom{n}{i}$ such vertices for each $i, 1 \leq i \leq n$. Hence, by (1) and (3) of Lemma 2.1, the result follows. □

Corollary 4.2. (1) *When $\alpha = -\frac{1}{2}$ in the general Randic connectivity Index, we obtained the Randic connectivity index.*
(2) *$R_1(H_T(n,k))$ represents the second Zagreb index.*
(3) *The modified Zagreb index is the index $R_{-1}(H_T(n,k))$.*
(4) *The sum connectivity index is $\chi_\alpha(H_T(n,k))$ when $\alpha = -\frac{1}{2}$.*
(5) *$\chi_1(H_T(n,k))$ is the first Zagreb index.*

Theorem 4.3. *The first general Zagreb index is*

$$M_\alpha(H_T(n,k)) = \begin{cases} \sum_{i=1}^{k-1} \binom{n}{i}\left[\binom{n-i}{k-i}\right]^\alpha + \binom{n}{k}\left(2^k + 2^{n-k} - 3\right)^\alpha \\ + \sum_{i=k+1}^{n} \binom{n}{i}\left(\binom{i}{k}\right)^\alpha \end{cases}$$

Proof. The result follows from Lemma 3.2 and using (2) of Lemma 2.1. □

Corollary 4.4. *The Forgotten index for the graph $H_T(n,k)$ is $M_3(H_T(n,k))$.*

5. Conclusion

In this paper, we determined formula for finding number of edges of the graph $H_T(n,k)$ for any positive integer $n>1$ and $1 \le k \le n-1$. The vertex connectivity and edge connectivity are identified as $k+1$ when $1 \le k \le \lfloor \frac{n}{2} \rfloor$, and as that of $H_T(n,n-k)$ when $\lfloor \frac{n}{2} \rfloor < k \le n-1$. Its clique number is 2 as well. We obtained formulae for finding number of vertex pairs with distances $1,2,3,$ and 4, and hence found that the diameter is 4. Some degree-based topological indices such as the general Randic connectivity index, the general sum connectivity index and the first Zagreb index were examined. Finding more topological indices that relate to this graph is intriguing.

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References

1. A. R. Ashrafi and A. Ghalavand, On edge-degree based invariants of graphs, *Asian-Eur. J. Math.*, **15**(9) (2022), Article ID: 2250161.
2. J. Bamberg, A. Devillers, M. Ioppolo and C. E. Praeger, Codes and designs in Johnson graphs from symplectic actions on quadratic forms, (2022), *arXiv:2202.06237.*
3. I. N. Cangul, A. Saleh, A. Alqesmah and H. Alashwali, Entire Wiener index of graphs, *Commun. Comb. Optim.*, **7**(2) (2022), 227–245.
4. S. R. Canoy Jr. and M. A. Labendia, Polynomial representation and degree sequence of a graph, *International Journal of Mathematical Analysis*, **8** (2014), 1445–1455.
5. G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw Hill Education, 2013.
6. Y. Chen and Y. Wang, On the diameter of generalized Kneser graphs, *Discrete Math.*, **308**(18) (2008), 4276–4279.
7. T. Chowdhury and A. Pramanik, LDPC code construction from bipartite Kneser graphs, *Proceedings of the 2nd International Conference on Communication*, Devices and Computing, Springer, (2020), 1-12.
8. C. Godsil and G. Royle, *Algebraic Graph Theory*, International series of monographs on Physics, Springer 2013.
9. I. Gutman, B. Furtula and V. Katanic, Randic index and information, *AKCE Int. J. Graphs Comb.*, **15**(3) (2018), 307–312.
10. R. Kogani and S. M. Mirafzal, On determining the distance spectrum of a class of distance integral graphs, *J. Algebr. Syst.*, **10**(2) (2023), 299–308.

11. A. E. Kupriyanov, A. V. Bobu and A. M. Raigorodskii, A generalization of Kneser graphs, *Math. Notes*, **107**(3) (2020), 392–403.
12. S. M. Mirafzal, The automorphism group of the bipartite Kneser graph, *Proceedings-Mathematical Sciences*, **129** (2018), 1–8.
13. J. Plesnik and S. Znam, On equality of edge-connectivity and minimum degree of a graph, *Arch. Math. (Brno)*, **25**(1) (1989), 19–25.
14. N. Saeed, K. Long, Z. S. Mufti, H. Sajid and A. Rehman, Degree-based topological indices of Boron B_{12}, *Journal of Chemistry*, **2021**(6) (2021), 1–6.
15. K. G. Sreekumar, K. Manilal and J. K. Rajan, 2S3 transformation for dyadic fractions in the interval $(0, 1)$, *Commun. Comb. Optim.*, **8**(2) (2023), 411–421.
16. K. G. Sreekumar, P. Ramesh Kumar and K. Manilal, Automorphism groups of bipartite Kneser type-k graphs, *Asian-Eur. J. Math.*, **16**(3) (2023), Article ID: 2350047.
17. M. Valencia-Pabon and J. C. Vera, On the diameter of Kneser graphs, *Discrete Math.*, **305**(1) (2005), 383–385.

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ON THE DIAMETER AND CONNECTIVITY OF BIPARTITE KNESER TYPE-K GRAPHS

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بررسی قطر و همبندی گراف‌های دو بخشی از نوع کنسر نوع-k

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فرض کنید $n \in \mathbb{Z}^+$، $n > ۱$ و k یک عدد صحیح باشد به طوری که $۱ \leq k \leq n - ۱$. گراف $H_T(n, k)$ به عنوان گرافی با مجموعه رأس V تعریف می‌شود که شامل تمام زیرمجموعه‌های ناتهی از $\mathcal{S}n = \{۱, ۲, ۳, \ldots, n\}$ است. این گراف، یک گراف دو بخشی با بخش‌های $(V_۱, V_۲)$ می‌باشد که در آن $V_۱$ شامل زیرمجموعه‌های k-عنصری از $\mathcal{S}n$ و $V_۲$ شامل زیرمجموعه‌های i-عنصری از $\mathcal{S}n$ است به طوری که $۱ \leq i \leq n,\ i \neq k$. یک یال بین رأس $U \in V_۱$ و رأس $W \in V_۲$ وجود دارد هرگاه $W \subset U$ یا $U \subset W$ باشد. در این مقاله، فرمول‌هایی برای تعداد یال‌ها، همبندی رأسی، همبندی یالی و چندجمله‌ای درجه ارائه شده است. سپس، عدد خوشه‌ای و قطر گراف $H_T(n, k)$ مورد بررسی قرار گرفته است. همچنین نشان داده شده که گراف‌های $H_T(n, k)$ و $H_T(n, n - k)$ یکریخت هستند. شاخص‌های توپولوژیکی مبتنی بر درجه همچون شاخص همبندی رندیک کلی، شاخص زاگرب کلی اول و شاخص همبندی مجموع کلی نیز محاسبه شده‌اند. به علاوه، اثبات شده است که گراف خطی $H_T(n, k)$ یک گراف دو بخشی نیست.

کلمات کلیدی: گراف‌های دو بخشی از نوع کنسر نوع-k، همبندی رأسی، عدد خوشه‌ای، قطر، شاخص همبندی رندیک کلی، شاخص زاگرب کلی اول، شاخص همبندی مجموع کلی.