Journal of Algebraic Systems, Vol. 13(No. 3): (2025), pp 15-27.
https://doi.org/10.22044/JAS.2023.13260.1737

ZERO FORCING NUMBER AND MAXIMUM NULLITY OF GENERAL POWER GRAPHS

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Abstract. Let $G=(V,E)$ be a simple and undirected graph. General power graph of G, shown by $\mathscr{P}_g(G)$, is a graph with the vertex set $\mathscr{P}(V(G))\setminus\phi$. Also two distinct vertices B and C are adjacent if and only if every $b\in B$ is adjacent to every $c\in C\setminus\{b\}$ in G. In this paper, we show that zero forcing number is equal to maximum nullity, for general power graphs of complete bipartite graphs.

1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $G=(V,E)$ to denote the graph with nonempty vertex set $V=V(G)$ and edge set $E=E(G)$. An edge of G with end vertices u and v is denoted by $u\sim v$. The *openneighbourhood* of a vertex denoted $N_G(v)$, is $\{u\in V(G): u\sim v\in E(G)\}$ and the *close neighbourhood* of vertex $v\in V(G)$, $N_G[v]$, is $N_G[v]=N_G(v)\cup\{v\}$. For a set $T\subseteq V(G)$, the *open neighbourhood* of T is $N_G(T)=\cup_{x\in T}N_G(x)$ and the closed neighbourhood of T is $N_G[T]=N_G(T)\cup T$. The *degree* of a vertex $x\in V(G)$ is $\deg_G(x)=\big|N_G(x)\big|$. The *minimum degree* and *maximum degree* of a graph G denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a set $S\subseteq V(G)$, the *induced subgraph* by S is denoted by $G[S]$. The length of the shortest cycle in a graph G is called girth of G and denoted by $girth(G)$. The notations P_n, C_n, K_n and $K_{r,s}$ are used for path, cycle, complete graph, and complete bipartite graph, respectively.

The set of symmetric matrices of graph G is defined by

$$S(G)=\{A\in S_n(\mathbb{R})\mid \mathcal{G}(A)=G\}.$$

The maximum nullity of G is

$$M(G)=\max\{null(A)\mid A\in S(G)\}$$

Published online: 9 April 2024
MSC(2020): Primary: 05C69; Secondary: 05C75.
Keywords: Power set; Zero forcing set; Maximum nullity.
Received: 18 June 2023, Accepted: 12 October 2023.
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and the minimum rank of G is

$$mr(G) = \min\{rank(A) \mid A \in S(G)\}.$$

Let each vertex of a graph G be given one of two colors "black" and "white". If a white vertex b is the only white neighbour of a black vertex a, then a changes the color of b to black. It is called (*color-change rule.*) Furthermore we say a forces b or b is forced by a.

Let B be the initial black vertices. Then B is said a *zero forcing* set of G if all of the vertices of G will be turned black after finitely many applications of the color-change rule. The *zero forcing number* of G, $Z(G)$, is the minimum cardinality among all zero forcing sets. The notation of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced by the "AIM Minimum Rank-Special Graphs Work Group" in (2008) [3]. They used the technique of zero forcing parameter of graph G and found an upper bound for the maximum nullity of G related to zero forcing sets. It is shown that for any graph G, $M(G) \leq Z(G)$. Also the following question has been raised in [3]. What is the class of graphs G for which $M(G) = Z(G)$? As a simple example, the complete graph K_n on n vertices has $Z(K_n) = M(K_n) = n-1$. In [8], Davila and Kenter conjectured the lower bound $Z(G) \geq (girth(G) - 2)(\delta(G) - 2) + 2$, for every graph G such that $girth(G) \geq 3$ and $\delta(G) \geq 2$. This conjecture was considered by Gentner [16], for $g = 4$ and for triangle free graphs. For more results, see [1], [8], [9], [7], [11], [12],[13], [14], [15],[17], [18], [21],[23].

Melody and Renson [19] in 2019, introduced the concept of power set graph of a simple graph. M. Eshaghi et al. [20] introduced the concept six types of power graphs related to a graph (or directed graph), with the help of set theory. They discussed the relation between Eulerian being the base graph and these six power graph types.

In this paper we rename one of six types of power graphs to *General Power Graph.* Also we show that, zero forcing number is equal to maximum nullity, for general power graphs of complete bipartite graphs.

2. General Power Graph

In this section, we introduce the general power graph. Then we will present some preliminary results for general power graphs.

Definition 2.1. Let G be a graph. *General power graph* of G, shown by $\mathscr{P}_g(G)$, is a graph with the vertex set $\mathscr{P}(V(G)) \setminus \phi$. Also two distinct vertices

B and C are adjacent if and only if every $b \in B$ is adjacent to every $c \in C \setminus \{b\}$ in G.

Example 2.2. The following is an example of a general power graph.

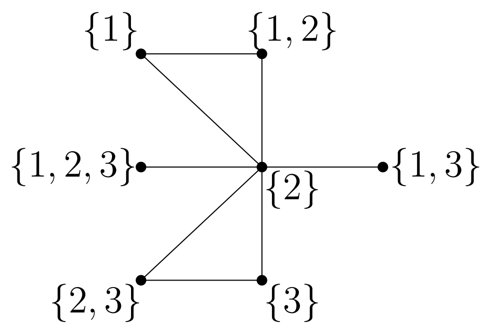

FIGURE 1. $\mathscr{P}_g(P_3)$

Theorem 2.3. *Let G be a graph, $X = \{x_1, \ldots, x_t\} \subseteq V(G)$. If induced subgraph on X in G does not have universal vertex and $N_G(x_1) \cap \ldots \cap N_G(x_t) = \emptyset$, then X is an isolated vertex in $\mathscr{P}_g(G)$.*

Proof. Let $Y \in N_{\mathscr{P}_g(G)}(X)$. Then every $y \in Y$ is adjacent to x_i in G for every $1 \leq i \leq t$. So $Y \subseteq N_G(x_1) \cap \ldots \cap N_G(x_t)$ or induced subgraph $G[X]$ has at least a universal vertex. However it is not true. Therefore X is an isolated vertex in $\mathscr{P}_g(G)$. □

Corollary 2.4. *Let G be a graph of order n. Then $\mathscr{P}_g(G)$ is a connected graph if and only if $\Delta(G) = n - 1$.*

Proof. Let u be a universal vertex of G. By Definition 2.1, the vertex $\{u\}$ is a universal vertex of $\mathscr{P}_g(G)$ and so $\mathscr{P}_g(G)$ is a connected graph.
Let G does not have any universal vertex and $X = V(G) = \{v_1, \ldots, v_n\}$. Then since $N_G(v_1) \cap \ldots \cap N_G(v_n) = \emptyset$, by Theorem 2.3, the vertex X is an isolated vertex in $\mathscr{P}_g(G)$. Therefore $\mathscr{P}_g(G)$ is not a connected graph. □

Theorem 2.5. *Let G be a graph, $X = \{x_1, \ldots, x_t\} \subseteq V(G)$, $N_G(x_1) \cap \ldots \cap N_G(x_t) = \emptyset$ and*

$$\mathscr{B} = \{x_i \in X \mid x_i \textit{ is a universal vertex in } G[X]\}.$$

If $\mathscr{B} \neq \emptyset$, then $N_{\mathscr{P}_g(G)}(X) = \mathscr{P}(\mathscr{B}) \setminus \{\emptyset, X\}$.

Proof. Clearly, $\mathscr{P}(\mathscr{B}) \setminus \{\emptyset, X\} \subseteq N_{\mathscr{P}_g(G)}(X)$. Let $Y \in N_{\mathscr{P}_g(G)}(X)$. Then $Y \neq X$ and every $y \in Y$ is adjacent to every vertex $x_i \in X \setminus \{y\}$. Since $N_G(x_1) \cap \ldots \cap N_G(x_t) = \emptyset$, so y is a universal vertex of $G[X]$. Hence $Y \in \mathscr{P}(\mathscr{B}) \setminus \{\emptyset, X\}$. Therefore $N_{\mathscr{P}_g(G)}(X) \subseteq \mathscr{P}(\mathscr{B}) \setminus \{\emptyset, X\}$. □

Definition 2.6. Let X and Y be two disjoint sets. For every $B \in \mathscr{P}(Y) \setminus \emptyset$, we define $B \vee \mathscr{P}(X) = \{B \cup A \mid A \in \mathscr{P}(X)\}$.

Theorem 2.7. *Let G be a graph, $X = \{x_1, \ldots, x_t\} \subseteq V(G)$, and $N_G(x_1) \cap \ldots \cap N_G(x_t) = X^*$.*

i) *If $t = 1$, then*
$$N_{\mathscr{P}_g(G)}(\{x_1\}) = (\mathscr{P}(N_G(x_1)) \setminus \emptyset) \cup (\{x_1\} \vee \mathscr{P}(N_G(x_1)) \setminus \emptyset).$$

ii) *If $t \geq 2$ and $C = \{x_i \in X \mid x_i$ is a universal vertex in $G[X]\}$, then $N_{\mathscr{P}_g(G)}(X) = \bigcup_{B \in \mathscr{P}(C) \setminus \emptyset} (B \vee \mathscr{P}(X^*)) \cup (\mathscr{P}(X^*) \setminus \emptyset)$.*

Proof. i) Let $X = \{x_1\}$. By Definition 2.1, we have
$$(\mathscr{P}(N_G(x_1)) \setminus \emptyset) \cup (\{x_1\} \vee \mathscr{P}(N_G(x_1)) \setminus \emptyset) \subseteq N_{\mathscr{P}_g(G)}(\{x_1\}).$$
Let $Y \in N_{\mathscr{P}_g(G)}(\{x_1\})$. Then every vertex $y \in Y$ is adjacent to vertex x_1 in graph G. Hence $Y \in (\mathscr{P}(N_G(x_1)) \setminus \emptyset) \cup (\{x_1\} \vee \mathscr{P}(N_G(x_1)) \setminus \emptyset)$. Therefore,
$$N_{\mathscr{P}_g(G)}(\{x_1\}) = (\mathscr{P}(N_G(x_1)) \setminus \emptyset) \cup (\{x_1\} \vee \mathscr{P}(N_G(x_1)) \setminus \emptyset).$$
ii) Let $Y \in N_{\mathscr{P}_g(G)}(X)$. Then every vertex $y \in Y$ is adjacent to vertex x_i in G, for every $1 \leq i \leq t$. So $y \in C$ or $y \in N_G(x_1) \cap \ldots \cap N_G(x_t) = X^*$. Hence $Y \in \bigcup_{B \in \mathscr{P}(C) \setminus \emptyset} (B \vee \mathscr{P}(X^*)) \cup (\mathscr{P}(X^*) \setminus \emptyset)$. Therefore,
$$N_{\mathscr{P}_g(G)}(X) \subseteq \bigcup_{B \in \mathscr{P}(C) \setminus \emptyset} (B \vee \mathscr{P}(X^*)) \cup (\mathscr{P}(X^*) \setminus \emptyset).$$
It is easy to see that every $Y \in \bigcup_{B \in \mathscr{P}(C) \setminus \emptyset} (B \vee \mathscr{P}(X^*)) \cup (\mathscr{P}(X^*) \setminus \emptyset)$ is adjacent to vertex X in general power graph $\mathscr{P}_g(G)$. Therefore
$$N_{\mathscr{P}_g(G)}(X) = \bigcup_{B \in \mathscr{P}(C) \setminus \emptyset} (B \vee \mathscr{P}(X^*)) \cup (\mathscr{P}(X^*) \setminus \emptyset).$$
□

Corollary 2.8. *The general power graph $\mathscr{P}_g(G)$ is a complete graph if and only if G is a complete graph.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. If $\mathscr{P}_g(G)$ is a complete graph, then for every i, $1 \leq i \leq n$, $\{v_i\}$ is a universal vertex of $\mathscr{P}_g(G)$. Thus for every i, $1 \leq i \leq n$, $\{v_i\}$ is adjacent to $\{v_1, \ldots, v_n\} \setminus \{v_i\}$ in $\mathscr{P}_g(G)$. By Definition 2.1, v_i is a universal vertex of G. Hence G is a complete graph.

Now suppose that G is a complete graph. For every vertex $X = \{x_1, \ldots, x_t\}$ in $\mathscr{P}_g(G)$, we have
$$X^* = N_G(x_1) \cap \ldots \cap N_G(x_t) = V(G) \setminus X.$$

Since every vertex of X is a universal vertex in graph G, so

$$(\mathscr{P}(X^*) \setminus \emptyset) \bigcup_{B\in\mathscr{P}(X)\setminus\emptyset}(B \vee \mathscr{P}(X^*)) = \mathscr{P}(V(G)) \setminus \{\emptyset, X\}.$$

By Theorem 2.7, $N_{\mathscr{P}_g(G)}(X) = \mathscr{P}(V(G)) \setminus \{\emptyset, X\}$. Therefore $\mathscr{P}_g(G)$ is a complete graph. □

3. Zero forcing number and maximum nullity of the general power graphs

In this Section, we show that zero forcing number is equal to maximum nullity, for general power graph of some graphs.

Theorem 3.1. [14] *Let G be a connected graph of order $n \geqslant 2$. Then $Z(G) = n - 1$ if and only if G is isomorphic to a complete graph of order n.*

Theorem 3.2. [3] *Let $G = (V, E)$ be a graph and $Z \subseteq V$ a zero forcing set for G. Then $M(G) \leqslant Z(G)$.*

The union of $G_i = (Vi, Ei)$, for $i = 1, \ldots, h$, is

$$\bigcup_{i=1}^h G_i = (\bigcup_{i=1}^h V_i, \bigcup_{i=1}^h E_i).$$

Theorem 3.3. [3] *If $G = \bigcup_{i=1}^h G_i$, then $mr(G) \leq \sum_{i=1}^h mr(G_i)$.*

Theorem 3.4. *Let G be a graph of order n with $V(G) = \{v_1, \ldots, v_n\}$. If for every i, $1 \leq i \leq t < n$, $deg_G(v_i) = n - t$ and for every j, $t + 1 \leq j \leq n$, $deg_G(v_j) = n - 1$, then $Z(\mathscr{P}_g(G)) = 2^n - t - 3$.*

Proof. Let $X = \{v_1, \ldots, v_t\}$ and $Y = \{v_{t+1}, \ldots, v_n\}$. Then

$$V(\mathscr{P}_g(G)) = \bigcup_{i=1}^{2^t-1}(B_i \vee \mathscr{P}(Y)) \cup (\mathscr{P}(Y) \setminus \emptyset),$$

where $B_i \in \mathscr{P}(X) \setminus \emptyset$ and $B_1 = \{v_1\}, \ldots, B_t = \{v_t\}$. By Definition 2.1, every vertex of $\mathscr{P}_g(G)$ in $\mathscr{P}(Y) \setminus \emptyset$ is a universal vertex of $\mathscr{P}_g(G)$. For every i, $1 \leq i \leq t$ induced subgraphs on $B_i \vee \mathscr{P}(Y)$ are isomorphic to $K_{2^{n-t}}$. For every i, $t + 1 \leq i \leq 2^t - 1$, induced subgraphs on $B_i \vee \mathscr{P}(Y)$ are isomorphic to $\overline{K_{2^{n-t}}}$. Also every vertex of $B_i \vee \mathscr{P}(Y)$ is not adjacent to the vertices of $B_j \vee \mathscr{P}(Y)$, where $i \neq j$ and $1 \leq i, j \leq 2^t - 1$. (See Fig. 2)

Let Z be a Zero forcing set of $\mathscr{P}_g(G)$ with minimum cardinality. If there are at least two white vertices in $\mathscr{P}(Y) \setminus \emptyset$, then every black vertex has at least two white vertices in its neighbourhood and so the forcing process is stopped. It is a contradiction. So $|Z \cap \mathscr{P}(Y) \setminus \emptyset| \geq 2^{n-t} - 2$.

If $|Z \cap (B_i \vee \mathscr{P}(Y))| \leq 2^{n-t} - 2$, for every i, $1 \leq i \leq t$, then every black vertex in $(B_i \vee \mathscr{P}(Y)) \cup (\mathscr{P}(Y) \setminus \emptyset)$ has at least two white vertices in its neighbourhood, which is not true. So $|Z \cap (B_i \vee \mathscr{P}(Y))| \geq 2^{n-t} - 1$, for every i, $1 \leq i \leq t$.

If there are at least two white vertices in $\bigcup_{i=t+1}^{2^t-1}(B_i \vee \mathscr{P}(Y))$, then they can forced only by the black vertices of $\mathscr{P}(Y) \setminus \emptyset$. But every black vertex of $\mathscr{P}(Y) \setminus \emptyset$ has at least two white vertices in its neighbourhood. Which is false. So $|\bigcup_{i=t+1}^{2^t-1}(B_i \vee \mathscr{P}(Y)) \cap Z| \geq 2^{n-t}(2^t - t - 1) - 1$. Therefore $|Z| \geq (2^n - 1) - (t + 2) = 2^n - t - 3$.

Now let $\mathscr{B} = \Big\{\{v_i\}\Big| 1 \leq i \leq t+1\Big\} \cup \Big\{\{v_1, v_2\}\Big\}$ and $\mathscr{L} = V(\mathscr{P}_g(G)) \setminus \mathscr{B}$ be the set of initial black vertices of $\mathscr{P}_g(G)$. Since $\{v_{t+1}\} \in \mathscr{P}(Y)$ is the only white neighbour of $\{v_1, v_2, v_{t+1}\} \in \{v_1, v_2\} \vee \mathscr{P}(Y)$, so $\{v_1, v_2, v_{t+1}\}$ forces $\{v_{t+1}\}$.

For every $1 \leq i \leq t$, $\{v_i\}$ is the only white neighbour of $\{v_i, v_{t+1}\} \in B_i \vee \mathscr{P}(Y)$, so $\{v_i\}$ is forced by $\{v_i, v_{t+1}\}$. Finally, $\{v_1, v_2\}$ is forced by $\{v_{t+1}\}$. Thus $\mathscr{L}$ is a zero forcing set of $\mathscr{P}_g(G)$. Hence

$$Z(\mathscr{P}_g(G)) \leq |\mathscr{L}| = (2^n - 1) - |\mathscr{B}| = (2^n - 1) - (t + 2) = (2^n - t - 3).$$

Therefore $Z(\mathscr{P}_g(G)) = 2^n - t - 3$.

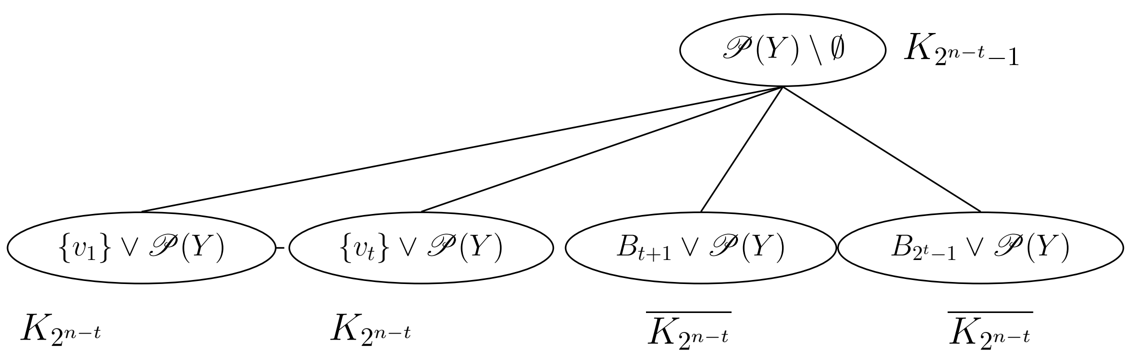

FIGURE 2.
Every vertex of $\mathscr{P}(Y) \setminus \emptyset$ is adjacent to every vertex of the other sets.

□

Corollary 3.5. *If $n \geq 2$ and $G \simeq K_{1,s}$, then*

$$Z(\mathscr{P}_g(G)) = 2^{(s+1)} - s - 3 = M(\mathscr{P}_g(G)).$$

Proof. Let $V(G) = \{a, x_1, \cdots, x_s\}$, $deg_G(a) = s$ and $X = \{x_1, \cdots, x_s\}$. Then

$$\begin{aligned} V(\mathscr{P}_g(G)) &= \Big\{\{x_i\}, \{a, x_i\} | 1 \le i \le s\Big\} \cup \{a\} \\ &\cup \Big\{B_\ell, B_\ell \cup \{a\} | B_\ell \in \mathscr{P}(X), |B_\ell| \ge 2\Big\}. \end{aligned}$$

Let A be the adjacency matrix of $\mathscr{P}_g(G)$ and $R_i(A)$ be the $i-t$h row of A such that for $1 \le i \le s$, $R_i(A)$ be a row corresponding to vertex of $\{x_i\}$. For $1 \le j \le s$, $R_{s+j}(A)$ is the row corresponding to vertex $\{a, x_j\}$. Also for $2s+1 \le \ell \le 2^{s+1}-2$, let $R_\ell(A)$ is the row corresponding to vertex B_ℓ or $B_\ell \cup \{a\}$, where $B_\ell \in \mathscr{P}(X)$ and $|B_\ell| \ge 2$. The row $R_{2^{s+1}-1}$ is corresponding to universal vertex $\{a\}$. (See the matrix $A(\mathscr{P}_g(K_{1,3}))$.

$$\mathrm{A}(\mathscr{P}_g(K_{1,3})) = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&0 \end{pmatrix}$$

Now let C be a matrix $(2^{s+1}-1) \times (2^{s+1}-1)$ such that

$$C_{ij} = \begin{cases} 1 & 1 \le i = j \le 2s \\ A_{ij} & o.w \end{cases}.$$

Let $R_i(C)$ be the $i-t$h row of C. Then we have,

$$\begin{aligned} R_1(C) &= R_{s+1}(C) \\ R_2(C) &= R_{s+2}(C) \\ &\vdots \\ R_s(C) &= R_{2s}(C). \end{aligned}$$

Also for every i, $2s+1 \leq i \leq 2^{s+1}-2$, we have, $R_{2s+1}(C) = R_i(C)$. Thus $rank(C) \leq s+2$. So $null(C) \geq (2^{s+1}-1)-(s+2) = 2^{s+1}-s-3$. (See matrix C for $A(\mathscr{P}_g(K_{1,3}))$.

$$C = \begin{pmatrix}
1&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&1&0&0&0&0&0&0&0&0&1\\
1&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&0
\end{pmatrix}$$

Hence $M(\mathscr{P}_g(G)) \geq 2^{s+1}-s-3$. By Theorem 3.2,

$$Z(\mathscr{P}_g(G)) \geq M(\mathscr{P}_g(G)) \geq 2^{s+1}-s-3.$$

By Theorem 3.4, $Z(\mathscr{P}_g(G)) = 2^{s+1}-s-3$. Therefore

$$Z(\mathscr{P}_g(G)) = M(\mathscr{P}_g(G)) = 2^{s+1}-s-3.$$

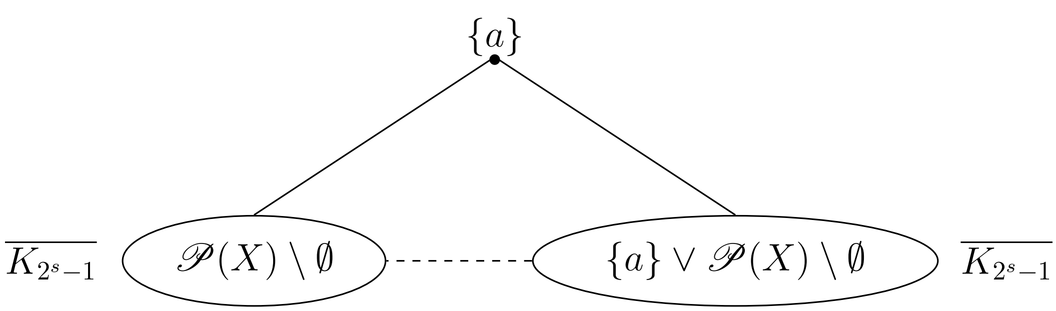

FIGURE 3. $\mathscr{P}_e(K_{1,s})$
Vertex $\{a\}$ is adjacent to every vertex of the other sets.
In $\mathscr{P}(X)\setminus\emptyset$ and $\{a\}\vee\mathscr{P}(X)\setminus\emptyset$, only and only the vertex $\{x_i\}$ is adjacent to $\{a, x_i\}$, for $1 \leq i \leq s$.

□

It is clear that $mr(K_{r,s}) = 2$. So $M(K_{r,s}) = r+s-2$. In the following theorem, we show that $Z(\mathscr{P}_g(K_{r,s})) = M(\mathscr{P}_g(K_{r,s}))$.

Theorem 3.6. *Let $r, s \geq 2$ and $G = K_{r,s}$. Then*

$$Z(\mathscr{P}_g(G)) = 2^{r+s} - 2r - 2s - 3 = M(\mathscr{P}_g(G)).$$

Proof. Let $X = \{x_i | 1 \leq i \leq r\}$ and $Y = \{y_i | 1 \leq i \leq s\}$ be two partitions of $K_{r,s}$. If $T = \Big\{\{x_i\} | 1 \leq i \leq r\Big\}$, $B_i \in \mathscr{P}(Y)$ and $|B_i| \geq 2$, then

$$\begin{aligned} V(\mathscr{P}_g(G)) = & \bigcup_{i=1}^{s} (\{y_i\} \vee (\mathscr{P}(X) \setminus \emptyset)) \\ & \bigcup_{i=1}^{2^s-s-1} (B_i \vee (\mathscr{P}(X) \setminus \emptyset)) \\ & \cup \mathscr{P}(X) \setminus (T \cup \{\emptyset\}) \cup T \cup (\mathscr{P}(Y) \setminus \emptyset). \end{aligned}$$

By Definition 2.1, for any two indices i and j, $(1 \leq i \leq s$, $1 \leq j \leq 2^s - s - 1)$, induced subgraphs on

$$\mathscr{P}(X) \setminus (T \cup \{\emptyset\}), T, (\mathscr{P}(Y) \setminus \emptyset),$$

$\{y_i\} \vee (\mathscr{P}(X) \setminus \emptyset)$ and $B_j \vee (\mathscr{P}(X) \setminus \emptyset)$ are empty graph, respectively. Also we have,

$$N_{\mathscr{P}_g(G)}(\{x_i\}) = \bigcup_{j=1}^{s} \Big\{\{x_i.y_j\}\Big\} \bigcup_{j=1}^{2^s-s-1} (B_j \cup \{x_i\}) \cup (\mathscr{P}(Y) \setminus \emptyset),$$

for every $1 \leq i \leq r$.

$$N_{\mathscr{P}_g(G)}(B_i \cup \{x_j\}) = \Big\{\{x_j\}\Big\},$$

for every $1 \leq i \leq 2^s - s - 1$ and every $1 \leq j \leq r$.
$N_{\mathscr{P}_g(G)}(\{y_i, x_j\}) = \Big\{\{x_j\}, \{y_i\}\Big\}$, for every i, j, where $1 \leq i \leq s$ and $1 \leq j \leq r$.
$N_{\mathscr{P}_g(G)}(\{y_i\}) = (\mathscr{P}(X) \setminus \emptyset) \cup (\{y_i\} \vee (\mathscr{P}(X) \setminus \emptyset))$, for every $1 \leq i \leq s$.
$N_{\mathscr{P}_g(G)}(A) = (\mathscr{P}(Y) \setminus \emptyset))$, for every $A \in \mathscr{P}(X) \setminus (T \cup \{\emptyset\})$.
$N_{\mathscr{P}_g(G)}(\{y_i\} \cup A) = \Big\{\{y_i\}\Big\}$, for every $A \in \mathscr{P}(X), |A| \geq 2$.
$N_{\mathscr{P}_g(G)}(B) = \mathscr{P}(X) \setminus \emptyset$, for every $B \in \mathscr{P}(Y), |B| \geq 2$.
$N_{\mathscr{P}_g(G)}(B_i \cup A) = \emptyset$, for every $A \in \mathscr{P}(X), |A| \geq 2$. (See Fig. 4)
It is easy to see that,

$$\mathscr{P}_g(G) = \bigcup_{t=1}^{s} (K_{1,2^r-1}) \bigcup_{t=1}^{r} (K_{1,2^s-1}) \bigcup_{t=1}^{(2^r-1-r)(2^s-1-s)} (K_1) \cup K_{2^r-1,2^s-1}.$$

By Theorem 3.3, $mr(\mathscr{P}_g(G)) \leq 2s + 2r + 2$. Hence,

$$M(\mathscr{P}_g(G)) \geq 2^{r+s} - 2s - 2r - 3.$$

By Theorem 3.2, $Z(\mathscr{P}_g(G)) \geq M(\mathscr{P}_g(G)) \geq 2^{r+s} - 2s - 2r - 3$. Now let

$$\mathscr{B} = \Big\{\{x_i\}, \{y_1, y_3, x_i\} | 1 \leq i \leq r\Big\} \cup \Big\{\{y_i\}, \{x_1, y_i\} | 1 \leq i \leq s\Big\} \cup \Big\{\{x_1, x_2\}, \{y_1, y_3\}\Big\}$$

and $Z = V(\mathscr{P}_g(G)) \setminus \mathscr{B}$ be the set of initial black vertices of $\mathscr{P}_g(G)$. For every i, $1 \leq i \leq r$, $\{x_i\}$ is the only white neighbour of $\{x_i, y_1, y_2\}$. So $\{x_i\}$ is forced by $\{x_i, y_1, y_2\}$. Since

$$N_{\mathscr{P}_g(G)}(\{y_1, y_2\}) = \mathscr{P}(X) \setminus \emptyset, \text{ so } \{x_1, x_2\}$$

is the only white neighbour of $\{y_1, y_2\}$. Thus $\{y_1, y_2\}$ forces $\{x_1, x_2\}$.

For every i, $1 \leq i \leq s$, we have $N_{\mathscr{P}_g(G)}(\{y_i, x_2\}) = \Big\{\{y_i\}, \{x_2\}\Big\}$. So $\{y_i\}$ is the only white neighbour of $\{y_i, x_2\}$. Hence for every i, $1 \leq i \leq s$, $\{y_i\}$ is forced by $\{y_i, x_2\}$. Since $\{y_1, y_3\}$ is the only white neighbour of $\{x_1, x_2\}$, so $\{y_1, y_3\}$ is forced by $\{x_1, x_2\}$.
For every i, $1 \leq i \leq s$, $\{y_i, x_1\}$ is the only white neighbour of $\{y_i\}$. So for every i, $1 \leq i \leq s$, $\{y_i\}$ forces $\{y_i, x_1\}$. Finally, for every $1 \leq i \leq r$, $\{x_i, y_1, y_3\}$ is the only white neighbour of $\{x_i\}$. So $\{x_i, y_1, y_3\}$ is forced by $\{x_i\}$. Hence Z is a zero forcing set of $\mathscr{P}_g(G)$. Thus

$$Z(\mathscr{P}_g(G)) \leq |Z| = 2^{r+s} - 2s - 2r - 3.$$

Therefore,

$$2^{r+s} - 2s - 2r - 3 \leq M(\mathscr{P}_g(G)) \leq Z(\mathscr{P}_g(G)) \leq 2^{r+s} - 2s - 2r - 3.$$

Hence, $Z(\mathscr{P}_g(G)) = M(\mathscr{P}_g(G)) = 2^{r+s} - 2s - 2r - 3$.

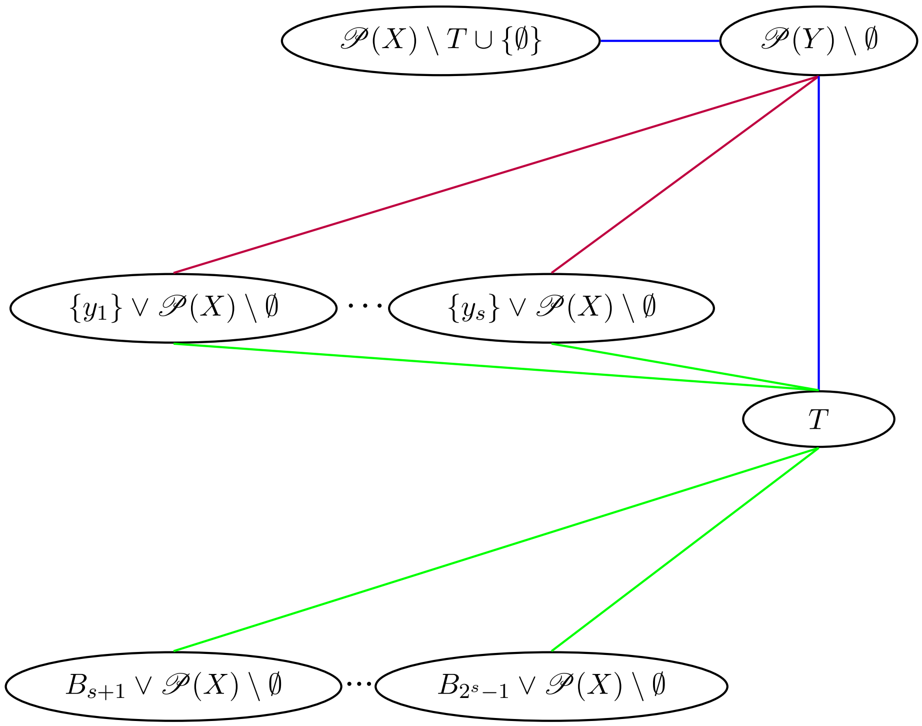

FIGURE 4. $\mathscr{P}_e(K_{r,s})$
Every vertex of $\mathscr{P}(Y)\setminus\emptyset$ is adjacent to every vertex of $\mathscr{P}(X)\setminus(T\cup\{\emptyset\})\cup T$. For every $1\le i\le s$ the vertex $\{y_i\}$ in $\mathscr{P}(Y)\setminus\emptyset$ is adjacent to every vertex of $\{y_i\}\vee(\mathscr{P}(X)\setminus\emptyset)$. For every $1\le i\le r$ the vertex $\{x_i\}$ in T is adjacent to the vertex $B_j\vee\{x_i\}$, where $1\le j\le 2^s-1$.

□

Acknowledgments

The authors are very grateful to the referee for his/her useful comments.

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Journal of Algebraic Systems

ZERO FORCING NUMBER AND MAXIMUM NULLITY OF GENERAL POWER GRAPHS

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عدد تحمیلی صفر و ماکسیمم پوچی گراف‌های توانی عمومی

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فرض کنید $G=(V,E)$ گراف ساده و غیر جهت‌دار باشد. گراف توانی عمومی گراف G که آن را با $\mathscr{P}_g(G)$ نشان می‌دهیم، گرافی با مجموعه رئوس $\mathscr{P}(V(G))\setminus\phi$ است. همچنین دو راس متمایز B و C در آن مجاورند اگر و تنها اگر هر $b\in B$ با هر $c\in C\setminus\{b\}$ در G مجاور باشد. در این مقاله نشان می‌دهیم برای گراف‌های توانی عمومی گراف‌های دو بخشی کامل، عدد تحمیلی صفر با ماکسیمم پوچی برابر است.

کلمات کلیدی: مجموعه توانی، عدد تحمیلی صفر گراف، ماکسیمم پوچی.