Journal of Algebraic Systems, Vol. 13(No. 3): (2025), pp 1-14.
https://doi.org/10.22044/JAS.2023.13263.1734

$\mathcal{K}$-FILTERS OF DISTRIBUTIVE LATTICES

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Abstract. The concept of $\mathcal{K}$-filters is introduced in distributive lattices and studied some properties of these classes of filters. Some necessary and sufficient conditions are derived for every π-filter of a distributive lattice to become a $\mathcal{K}$-filter. Some equivalent conditions are derived for every D-filter of a distributive lattice to become a $\mathcal{K}$-filter. Quasi-complemented lattices are characterized with the help of $\mathcal{K}$-filters.

Introduction

Many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T. P. Speed [14] and W. H. Cornish [4] made an extensive study of annihilators in distributive lattices. In [5], some properties of minimal prime filters are studied in distributive lattices and the properties of dense elements and D-filters are studied in MS-algebras [9]. In [2], the notion of D-filters was introduced in pseudo-complemented semilattices. Later it was generalized by the author [9] in MS-algebras. In the note [10], the authors introduced the concepts of dual annihilators and μ-filters in distributive lattices. Certain topological properties of prime μ-filters are also investigated in this paper. In [11], the author introduced the notion of normal filters and characterized the quasi-complemented lattices with the help of normal filters. In [8], the authors investigated certain important properties of prime D-filters of distributive lattices. Recently in 2022, M. S. Rao and Ch. V. Rao introduced the concept of ω-filters [13] in distributive lattices and characterized the ω-filters with the help of minimal prime D-filters. In [12], the authors introduced the concepts of regular filters and π-filters of distributive lattices.

The main aim of this paper is to characterize quasi-complemented lattices with the help of special kind of D-filters of distributive lattices. The notion of $\mathcal{K}$-filters is introduced and investigated certain properties of these filters with the help of maximal filters and minimal prime D-filters of distributive

Published online: 9 April 2024
MSC(2010): 06D99.
Keywords: D-filter; Regular filter; π-Filter; ω-Filter; $\mathcal{K}$-Filter; Quasi-complemented lattice.
Received: 15 June 2023, Accepted: 19 August 2023.

lattices. It is initially characterize quasi-complemented lattices with the help its prime D-filters. It is observed that every $\mathcal{K}$-filter of a distributive lattice is a π-filter. A set of equivalent conditions is given for every π-filter of a distributive lattice to become a $\mathcal{K}$-filter. It is again observed that every proper $\mathcal{K}$-filter of a distributive lattice is an ω-filter but not the converse. However, some equivalent conditions are derived for every ω-filter of a distributive lattice to become a $\mathcal{K}$-filter. Some equivalent conditions are derived for the class of all $\mathcal{K}$-filters of a distributive lattice to become a sublattice of the lattice of all filters which leads to a characterization of quasi-complemented lattices. Another characterization theorem of quasi-complemented lattices is given which shows that every D-filter of a quasi-complemented lattice to become a $\mathcal{K}$-filter. Finally, the class of all Boolean algebras are characterized with the help of $\mathcal{K}$-filters of distributive lattices.

1. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [3], [10], [8], [13], [12] and [14].

Definition 1.1. [1] A type $(2,2)$ algebraic structure $(L,\wedge,\vee)$ is called a distributive lattice if it satisfies the following properties $(1),(2),(3)$ and (4) along with (5) or $(5')$:

(1) $x\wedge x=x$, $x\vee x=x$,
(2) $x\wedge y=y\wedge x$, $x\vee y=y\vee x$,
(3) $(x\wedge y)\wedge z=x\wedge(y\wedge z)$, $(x\vee y)\vee z=x\vee(y\vee z)$,
(4) $(x\wedge y)\vee x=x$, $(x\vee y)\wedge x=x$,
(5) $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$,
(5′) $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$,

for all $x,y,z\in L$.

If L is a lattice, then the ideal (resp. filter) A of L is a non-empty lower segment (resp. upper segment) closed under the operation $\vee$ (resp. operation $\wedge$). For any lattice L with smallest element 0, the set $\mathcal{I}(L)$ of all ideals of L forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of $(L,\vee,\wedge,1)$ forms a complete distributive lattice. A proper ideal (resp. filter) I of a lattice L is said to be *maximal* if there exists no proper ideal (resp. filter) J such that $I\subset J$. For any element a of a lattice L, the *principal ideal generated by* a is the set $(a]=\{x\in L\mid x\leq a\}$. The set of all principal ideals of a lattice L is a sublattice of $\mathcal{I}(L)$. Dually the set $[a)=\{x\in L\mid a\leq x\}$

is called a *principal filter* generated by a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. A proper ideal (resp. proper filter) P of a lattice L is called *prime* if for all $a, b \in L$, $a \wedge b \in P$ ($a \vee b \in P$) then $a \in P$ or $b \in P$. Every maximal ideal (resp. maximal filter) is prime.

The *annihilator* [14] of an element a of a distributive lattice L, is given as the set $(a)^* = \{ x \in L \mid x \wedge a = 0 \}$. By a *dense element* of a lattice L, we mean an element x such that $(x)^* = \{0\}$. In a distributive lattice L, the set $D(L)$ of all dense elements forms a filter of L. A distributive lattice L with 0 is called *quasi-complemented* [7] if for each $x \in L$ there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D(L)$.

Definition 1.2. [10] For any subset A of a distributive lattice L, the set A^+ is define as $A^+ = \{x \in L \mid a \vee x = 1 \text{ for all } a \in A\}$.

Clearly A^+ is a filter of any distributive lattice L and A^+ is known as the *dual annihilator* of the set A. For brevity, we denote $\{a\}^+$ by $(a)^+$. It can be seen immediately that $(a)^+ = L$ if and only if $a = 1$.

Definition 1.3. [8] A filter F of a lattice L is called a D-filter if $D(L) \subseteq F$.

In any distributive lattice L, it is clear that $D(L)$ is the smallest D-filter of the lattice L. For any subset A of a distributive lattice L, define $A^\circ = \{x \in L \mid a \vee x \in D \text{ for all } a \in A\}$. In case of $A = \{a\}$, we simply represent $(\{a\})^\circ$ by $(a)^\circ$. Then it is obvious that $(1)^\circ = L$. Obviously, $L^\circ = D(L)$ and $D(L)^\circ = L$. Further, $D(L) \subseteq A^\circ$ for any subset A of a lattice L. For any subset A of L, A° is a D-filter of L. For any $x \in L$, it is obvious that $([x))^\circ = (x)^\circ$. Then $(0)^\circ = D(L)$.

Proposition 1.4. [8] *Let L be a distributive lattice and $a, b \in L$. Then*

(1) *$a \leq b$ implies $(a)^\circ \subseteq (b)^\circ$,*
(2) *$(a \wedge b)^\circ = (a)^\circ \cap (b)^\circ$,*
(3) *$(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ}$,*
(4) *$(a)^\circ = L$ if and only if $a \in D(L)$.*

Suppose that F is a D-filter and P a prime D-filter of a distributive lattice L such that $F \subseteq P$. Then P is called a *minimal prime D-filter belonging to F* if there is no prime D-filter Q such that $F \subseteq Q \subset P$. A prime D-filter belonging to D is simply called *minimal prime D-filter*. A prime D-filter P of a lattice L is minimal [8] if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in D(L)$.

A filter F of a lattice L is called a regular [12] if $F = F^{\circ\circ}$. Clearly, each $(x)^\circ$

is a regular filter. A filter F of a lattice F is called a π-filter [12] if $(x)^{\circ\circ} \subseteq F$ whenever $x \in F$. Every regular filter of a distributive lattice is a π-filter. For any ideal I of a lattice L, define $\omega(I) = \{x \in L \mid x \vee a \in D(L)$ for some $a \in I\}$. In [13], it is observed that $\omega(I)$ is a D-filter of L. A filter F of a lattice L is called an ω-filter if $F = \omega(I)$ for some ideal I of L. Every minimal prime D-filter of L is an ω-filter. Throughout this article, all lattices are bounded distributive lattices unless otherwise mentioned.

2. Main results

In this section, the concept of $\mathcal{K}$-filters is introduced in lattices. Equivalency between $\mathcal{K}$-filters and ω-filters of lattices is established. A set of equivalent conditions is derived for every filter of a lattice to become a $\mathcal{K}$-filter.

Lemma 2.1. *Every maximal filter of a lattice is a prime D-filter.*

Proof. Let M be a maximal filter of a lattice L and $x \in D(L)$. Clearly M is prime. Suppose $x \notin M$. Since M is maximal, we get $M \vee [x) = L$. Thus $0 \in M \vee [x)$. Hence there exists $0 \neq m \in M$ such that $m \wedge x = 0$. Thus $m \in (x)^* = \{0\}$, which is a contradiction. Hence $x \in M$, which concludes that $D(L) \subseteq M$. Therefore M is a prime D-filter of L. □

Theorem 2.2. *The following assertions are equivalent in a lattice L:*

(1) *L is quasi-complemented;*
(2) *every prime D-filter is maximal;*
(3) *every prime D-filter is minimal.*

Proof. (1) $\Rightarrow$ (2): Assume that L is quasi-complemented. Let P be a prime D-filter of L. Suppose there exists a proper filter Q such that $P \subset Q$. Choose $x \in Q - P$. Since L is quasi-complemented, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D(L)$. Since $x \notin P$, we get $(x)^\circ \subseteq P$. Hence $y \in (x)^\circ \subseteq P \subset Q$. Thus $0 = x \wedge y \in Q$, which is a contradiction. Therefore P is maximal.

(2) $\Rightarrow$ (3): Since every maximal filter is a prime D-filter, it is clear.

(3) $\Rightarrow$ (1): Assume that every prime D-filter is minimal. Let $x \in L$. Suppose $[x) \vee (x)^\circ \neq L$. Then there exists a prime D-filter P of L such that $[x) \vee (x)^\circ \subseteq P$. Then $x \in P$ and $(x)^\circ \subseteq P$. Since P is minimal and $(x)^\circ \subseteq P$, we get $x \notin P$ which is a contradiction. Hence $[x) \vee (x)^\circ = L$. Thus $0 \in [x) \vee (x)^\circ$. Then there exist $b \in (x)^\circ$ such that $x \wedge b = 0$. Since $b \in (x)^\circ$, we get $b \vee x \in D(L)$. Therefore L is quasi-complemented. □

Definition 2.3. For any filter F of a lattice L, define $\mathcal{K}(F)$ as follows:

$$\mathcal{K}(F) = \{x \in X \mid (x)^{\circ} \vee F = L\}.$$

Clearly $\mathcal{K}(L) = L$. For $F = D(L)$, obviously $\mathcal{K}(D(L)) = D(L)$.

Lemma 2.4. *For any filter F of a lattice L, $\mathcal{K}(F)$ is a D-filter of L.*

Proof. Clearly $D(L) \subseteq \mathcal{K}(F)$. Let $x, y \in \mathcal{K}(F)$. Then, we get that $(x)^{\circ} \vee F = L$ and $(y)^{\circ} \vee F = L$. Hence

$$(x \wedge y)^{\circ} \vee F = \big\{(x)^{\circ} \cap (y)^{\circ}\big\} \vee F = \big\{(x)^{\circ} \vee F\big\} \cap \big\{(y)^{\circ} \vee F\big\} = L \cap L = L.$$

Hence $x \wedge y \in \mathcal{K}(F)$. Let $x \in \mathcal{K}(F)$ and $x \leq y$. Then $(x)^{\circ} \subseteq (y)^{\circ}$ and thus $L = (x)^{\circ} \vee F \subseteq (y)^{\circ} \vee F$. Hence $y \in \mathcal{K}(F)$. Therefore $\mathcal{K}(F)$ is a D-filter of L. □

In the following, some elementary properties of $\mathcal{K}(F)$ are derived.

Lemma 2.5. *For any two filters F and G of a lattice L, we have*

(1) *$D(L) \subseteq F$ if and only if $\mathcal{K}(F) \subseteq F$,*
(2) *$F \subseteq G$ implies $\mathcal{K}(F) \subseteq \mathcal{K}(G)$,*
(3) *$\mathcal{K}(F \cap G) = \mathcal{K}(F) \cap \mathcal{K}(G)$,*
(4) *$\mathcal{K}(F) \vee \mathcal{K}(G) \subseteq \mathcal{K}(F \vee G)$.*

Proof. (1) Assume that $D(L) \subseteq F$. Let $x \in \mathcal{K}(F)$. Then, we get $(x)^{\circ} \vee F = L$. Hence $x \in (x)^{\circ} \vee F$. Thus $x = a \wedge b$ for some $a \in (x)^{\circ}$ and $b \in F$. Since $a \in (x)^{\circ}$, we get $a \vee x \in D(L)$. Then there exists some $d \in D(L)$ such that $a \vee x = d$. Thus

$$x = x \vee x = (a \wedge b) \vee x = (a \vee x) \wedge (b \vee x) = d \wedge (b \vee x) \in D(L) \vee F = F$$

because of $b \vee x \in F$. Therefore $\mathcal{K}(F) \subseteq F$. Converse follows immediately due to $D(L) \subseteq \mathcal{K}(F)$.

(2) Suppose $F \subseteq G$. Let $x \in \mathcal{K}(F)$. Then $L = (x)^{\circ} \vee F \subseteq (x)^{\circ} \vee G$. Therefore $x \in \mathcal{K}(G)$.

(3) Clearly $\mathcal{K}(F \cap G) \subseteq \mathcal{K}(F) \cap \mathcal{K}(G)$. Conversely, let $x \in \mathcal{K}(F) \cap \mathcal{K}(G)$. Then $(x)^{\circ} \vee F = (x)^{\circ} \vee G = L$. Now

$$(x)^{\circ} \vee (F \cap G) = \{(x)^{\circ} \vee F\} \cap \{(x)^{\circ} \vee G\} = L \cap L = L.$$

Hence $x \in \mathcal{K}(F \cap G)$. Thus $\mathcal{K}(F) \cap \mathcal{K}(G) \subseteq \mathcal{K}(F \cap G)$. Therefore $\mathcal{K}(F \cap G) = \mathcal{K}(F) \cap \mathcal{K}(G)$.

(4) It follows from (2). □

Definition 2.6. A filter F of a lattice L is called a $\mathcal{K}$-*filter* if $F = \mathcal{K}(F)$.

Clearly $D(L)$ and L are $\mathcal{K}$-filters of L. In [13], the class of all π-filters of a lattice L is characterized in terms of D-annulets of the lattice. In the following theorem, it is proved that the class of all π-filters of a lattice L contains properly the class of all $\mathcal{K}$-filters of L.

Proposition 2.7. *Every $\mathcal{K}$-filter of a lattice is a π-filter.*

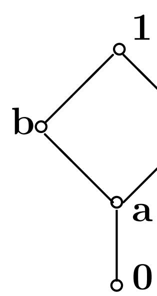
Proof. Let F be a $\mathcal{K}$-filter of a lattice L. Then $\mathcal{K}(F) = F$. Let $x \in F$. Then $(x)^\circ \vee F = L$. Now, let $t \in (x)^{\circ\circ}$. Then $(x)^\circ \subseteq (t)^\circ$. Hence

$$L = (x)^\circ \vee F \subseteq (t)^\circ \vee F.$$

Thus $t \in \mathcal{K}(F) = F$, which concludes that $(x)^{\circ\circ} \subseteq F$. Therefore F is a π-filter of L. □

The converse of Proposition 2.7 is not true. i.e. every π-filter of a lattice need not be a $\mathcal{K}$-filter. It can be seen in the following example:

Example 2.8. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure.



Consider the filter $F = \{b, 1\}$. It can be easily observed that $(b)^{\circ\circ} \subseteq F$. Hence F is a π-filters of L. Observe that $(b)^\circ \vee F = \{a, b, c, 1\} \neq L$. Therefore F is not a $\mathcal{K}$-filter of L.

However, in the following theorem, some equivalent conditions are given for every regular filter of a lattice to become a $\mathcal{K}$-filter.

Theorem 2.9. *The following assertions are equivalent in a lattice L:*

(1) *every π-filter is a $\mathcal{K}$-filter;*
(2) *every regular filter is a $\mathcal{K}$-filter;*
(3) *for each $x \in L$, $(x)^{\circ\circ}$ is a $\mathcal{K}$-filter;*
(4) *for each $x \in L$, $(x)^\circ \vee (x)^{\circ\circ} = L$.*

Proof. (1) ⇒ (2): Since every regular filter is a π-filter, it is clear.
(2) ⇒ (3): Since each $(x)^{\circ\circ}$ is a regular filter, it is clear.

(3) $\Rightarrow$ (4): Let $x \in L$. Since $(x)^{\circ\circ}$ is a $\mathcal{K}$-filter of L, we get $(x)^{\circ\circ} = \mathcal{K}\big((x)^{\circ\circ}\big)$. Clearly $x \in (x)^{\circ\circ} = \mathcal{K}\big((x)^{\circ\circ}\big)$. Hence $(x)^{\circ} \vee (x)^{\circ\circ} = L$.

(4) $\Rightarrow$ (1): Assume that $(x)^{\circ} \vee (x)^{\circ\circ} = L$ for each $x \in L$. Let F be a π-filter of L. Clearly $\mathcal{K}(F) \subseteq F$. Conversely, let $x \in F$. Since F is a π-filter, we get $(x)^{\circ\circ} \subseteq F$. Hence $L = (x)^{\circ} \vee (x)^{\circ\circ} \subseteq (x)^{\circ} \vee F$. Thus $x \in \mathcal{K}(F)$. Therefore F is a $\mathcal{K}$-filter of L. □

In [13], authors studied the properties of ω-filters and proved that every ω-filter of a lattice is the intersection of all minimal prime D-filters containing it. In the following, it is proved that the class of all $\mathcal{K}$-filters is properly contained in the class of all ω-filters.

Theorem 2.10. *Every proper $\mathcal{K}$-filter of a lattice is an ω-filter.*

Proof. Let F be a proper $\mathcal{K}$-filter of a lattice L. Then $\mathcal{K}(F) = F$. Consider $S = \{\ x \in L \mid (x)^{\circ\circ} \vee F = L\ \}$. We first show that S is an ideal of L such that $S \cap D = \emptyset$. Clearly $0 \in S$. Let $x, y \in S$. Then

$$\begin{aligned}(x \vee y)^{\circ\circ} \vee F &= \big\{(x)^{\circ\circ} \cap (y)^{\circ\circ}\big\} \vee F \\ &= \big\{(x)^{\circ\circ} \vee F\big\} \cap \big\{(y)^{\circ\circ} \vee F\big\} \\ &= L \cap L \\ &= L.\end{aligned}$$

Hence $x \vee y \in S$. Let $x \in S$ and $y \leq x$. Then $L = (x)^{\circ\circ} \vee F \subseteq (y)^{\circ\circ} \vee F$. Hence $y \in S$. Thus S is an ideal of L. Suppose $x \in S \cap D(L)$. Then $(x)^{\circ\circ} \vee F = L$ and $(x)^{\circ\circ} = D(L)$. Hence $F = D(L) \vee F = L$, which is a contradiction. Hence $S \cap D(L) = \emptyset$. We now show that $F = \omega(S)$. Let $x \in \omega(S)$. Then $x \vee y \in D(L)$ for some $y \in S$. Now

$$\begin{aligned}x \vee y \in D(L) &\Rightarrow y \in (x)^{\circ} \\ &\Rightarrow (y)^{\circ\circ} \subseteq (x)^{\circ} \\ &\Rightarrow L = (y)^{\circ\circ} \vee F \subseteq (x)^{\circ} \vee F \quad \text{since } y \in S \\ &\Rightarrow x \in \mathcal{K}(F) = F \qquad \text{since } F \text{ is a } \mathcal{K}\text{-filter}\end{aligned}$$

which concludes that $\omega(S) \subseteq F$. Conversely, let $x \in F = \mathcal{K}(F)$. Then $(x)^{\circ} \vee \mathcal{K}(F) = L$. Therefore $0 \in (x)^{\circ} \vee \mathcal{K}(F)$. Hence $0 = a \wedge b$ for some

$a \in (x)^\circ$ and $b \in \mathcal{K}(F)$. Thus $a \vee x \in D(L)$ and $(b)^\circ \vee F = L$. Now

$$\begin{aligned} a \wedge b = 0 \Rightarrow\ & (a \wedge b)^\circ = (0)^\circ = D(L) \\ \Rightarrow\ & (a)^\circ \cap (b)^\circ = D(L) \\ \Rightarrow\ & (b)^\circ \subseteq (a)^{\circ\circ} \\ \Rightarrow\ & L = (b)^\circ \vee F \subseteq (a)^{\circ\circ} \vee F \quad \text{since } b \in \mathcal{K}(F) \\ \Rightarrow\ & a \in S \text{ and } a \vee x \in D(L) \\ \Rightarrow\ & x \in \omega(S) \end{aligned}$$

which gives $F = \mathcal{K}(F) \subseteq \omega(S)$. Hence $F = \omega(S)$. Therefore F is an ω-filter of L. □

The converse of Theorem 2.10 is not true. i.e. every ω-filter of a lattice need not be a $\mathcal{K}$-filter. For, consider the distributive lattice given in Example 2.8. Consider $F = \{1, b\}$ and $I = \{0, a, c\}$. Clearly F is a filter and I is an ideal of L such that $F = \omega(I)$. Hence F is an ω-filter of L. Now, observe that $\mathcal{K}(F) = \{1\}$, because of $(b)^\circ \vee F = \{1, a, b, c\} \neq L$. Therefore F is not a $\mathcal{K}$-filter of L.

Proposition 2.11. *For each $a \in L - D$, $(a)^\circ$ is an ω-filter of L.*

Proof. Let $a \in L - D(L)$. Clearly $(a] \cap D(L) = \emptyset$. We show that $(a)^\circ = \omega((a])$. Let $x \in (a)^\circ$. Then $x \vee a \in D(L)$. Since $a \in (a]$, we get $x \in \omega((a])$. Hence $(a)^\circ \subseteq \omega((a])$. Conversely, let $x \in \omega((a])$. Then $x \vee t \in D(L)$ for some $t \in (a]$. Since $x \vee t \leq x \vee a$, we get $x \vee a \in D(L)$. Hence $x \in (a)^\circ$. Therefore $\omega((a]) \subseteq (a)^\circ$. □

Proposition 2.12. *Every prime $\mathcal{K}$-filter is a minimal prime D-filter.*

Proof. Let P be a prime $\mathcal{K}$-filter of a lattice L. Then $P = \mathcal{K}(P)$. Let $x \in P$. Since $x \in \mathcal{K}(P)$, we get $(x)^\circ \vee P = L$. Hence $0 \in (x)^\circ \vee P$. Thus there exist $a \in (x)^\circ$ and $b \in P$ such that $a \wedge b = 0$. Since $a \in (x)^\circ$, we get $a \vee x \in D(L)$. Suppose $a \in P$. Then $0 = a \wedge b \in P$, which is a contradiction. Thus to each $x \in P$, there exists $a \notin P$ such that $x \vee a \in D(L)$. By Lemma (2.4), P is a minimal prime D-filter of L. □

The converse of Proposition 2.12 is not true. For, consider the minimal prime D-filter $P = \{1, a\}$ of the distributive lattice given in Example 2.8. Observe that $\mathcal{K}(P) = \{1\}$, because of $(a)^\circ \vee P = \{1, a, b, c\} \neq L$. Therefore P is not a $\mathcal{K}$-filter of L. However, in the following theorem, a set of equivalent conditions is established for every minimal prime D-filter of a lattice to become a prime $\mathcal{K}$-filter.

Theorem 2.13. *The following assertions are equivalent in a lattice L:*

(1) *Every minimal prime D-filter is a prime $\mathcal{K}$-filter;*
(2) *for each $x \in L$, $(x)^\circ \vee (x)^{\circ\circ} = L$;*
(3) *every ω-filter is a $\mathcal{K}$-filter;*
(4) *every prime ω-filter is a $\mathcal{K}$-filter.*

Proof. (1) $\Rightarrow$ (2): Assume that every minimal prime D-filter is a prime $\mathcal{K}$-filter. Let $x \in L$. Suppose $(x)^\circ \vee (x)^{\circ\circ} \neq L$. Then there exists a maximal ideal M such that $\{(x)^\circ \vee (x)^{\circ\circ}\} \cap M = \emptyset$. Since $D(L) \subseteq (x)^\circ \vee (x)^{\circ\circ}$, we get $M \cap D(L) = \emptyset$. Hence $L - M$ is a minimal prime D-filter of L. By the assumption, $L - M$ is a $\mathcal{K}$-filter. Suppose $x \in M$. Since $x \in (x)^{\circ\circ}$, we get $x \in \{(x)^\circ \vee (x)^{\circ\circ}\} \cap M$ which is a contradiction. Thus $x \notin M$ and therefore $x \in L - M = \mathcal{K}(L - M)$. Hence $(x)^\circ \vee (L - M) = L$, which gives that $0 \in (x)^\circ \vee (L - M)$. Then $a \wedge b = 0 \in M$ for some $a \in (x)^\circ$ and $b \in L - M$. Since $b \notin M$ and M is prime, we must have $a \in M$. Hence $a \in \{(x)^\circ \vee (x)^{\circ\circ}\} \cap M$, which is a contradiction. Therefore $(x)^\circ \vee (x)^{\circ\circ} = L$ for all $x \in L$.

(2) $\Rightarrow$ (3): Let F be an ω-filter of L. Clearly $\mathcal{K}(F) \subseteq F$. Conversely, let $x \in F$. Since F is an ω-filter, we get $(x)^{\circ\circ} \subseteq F$. Hence $L = (x)^\circ \vee (x)^{\circ\circ} \subseteq (x)^\circ \vee F$. Thus $x \in \mathcal{K}(F)$. Therefore F is a $\mathcal{K}$-filter of L.

(3) $\Rightarrow$ (4): is clear.

(4) $\Rightarrow$ (1): Since every minimal prime D-filter is a prime ω-filter, it is obvious. $\square$

Definition 2.14. For any proper filter F of a lattice L, define
$$\Omega(F) = \{x \in L \mid (x)^\circ \nsubseteq F\}.$$

Proposition 2.15. *Let L be a lattice and M be a maximal filter of L. Then the set $\Omega(M)$ is a D-filter of L such that $\Omega(M) \subseteq M$.*

Proof. Let M be a maximal filter. Clearly $D(L) \subseteq M$. Since M is proper, we get $(d)^\circ \nsubseteq M$ for any $d \in D(L)$. Hence $D(L) \subseteq \Omega(M)$. Suppose $x, y \in \omega(M)$. Then $(x)^\circ \nsubseteq M$ and $(y)^\circ \nsubseteq M$. Hence $M \subset M \vee (x)^\circ$ and $M \subset M \vee (y)^\circ$. Since M is maximal, we get $M \vee (x)^\circ = L$ and $M \vee (y)^\circ = L$. Thus, we get
$$\begin{aligned} M \vee (x \wedge y)^\circ &= M \vee \big\{(x)^\circ \cap (y)^\circ\big\} \\ &= \big\{M \vee (x)^\circ\big\} \cap \big\{M \vee (y)^\circ\big\} \\ &= L \cap L \\ &= L. \end{aligned}$$

If $(x \wedge y)^\circ \subseteq M$, then $M = L$ which is a contradiction. Hence $(x \wedge y)^\circ \nsubseteq M$. Thus $x \wedge y \in \Omega(M)$. Again, let $x \in \Omega(M)$ and $x \leq y$. Then $(x)^\circ \nsubseteq M$ and $x \leq y$. Since $x \leq y$, we get $(x)^\circ \subseteq (y)^\circ$. Hence $(y)^\circ \nsubseteq M$. Hence $y \in \Omega(M)$. Therefore $\Omega(M)$ is a D-filter of L. Now, let $x \in \Omega(M)$. Then $(x)^\circ \nsubseteq M$. Hence, there exists $a \in (x)^\circ$ such that $a \notin M$. Since $a \in (x)^\circ$, we get $a \vee x \in D(L)$. Hence $[a \vee x) \subseteq D(L)$. Suppose $x \notin M$. Then $M \vee [x) = L$. Since $a \notin M$, we get $M \vee [a) = L$. Hence

$$L = M \vee \{[a) \cap [x)\} = M \vee [a \vee x) \subseteq M \vee D(L) = M,$$

which is a contradiction. Hence $x \in M$. Therefore $\Omega(M) \subseteq M$. □

Proposition 2.16. *Let P be a prime D-filter of a lattice L. Then*

(1) *$\mathcal{K}(P) \subseteq \Omega(P)$,*
(2) *if P is maximal, then $\mathcal{K}(P) = \Omega(P)$.*

Proof. (1) Let $x \in \mathcal{K}(P)$. Then $(x)^\circ \vee P = L$. Suppose $(x)^\circ \subseteq P$. Then $P = L$, which is a contradiction. Hence $(x)^\circ \nsubseteq P$. Thus $x \in \Omega(P)$. Therefore $\mathcal{K}(P) \subseteq \Omega(P)$.

(2) From (1), we get $\mathcal{K}(P) \subseteq \Omega(P)$. Conversely, let $x \in \Omega(P)$. Then $(x)^\circ \nsubseteq P$. Since P is maximal, we get $(x)^\circ \vee P = L$. Thus $x \in \mathcal{K}(P)$. Therefore $\Omega(P) = \mathcal{K}(P)$. □

Let us denote that μ is the set of all maximal filters of a lattice L. For any filter F of a lattice L, we also denote $\mu(F) = \{M \in \mu \mid F \subseteq M\}$.

Theorem 2.17. *For any filter F of a lattice L, $\mathcal{K}(F) = \bigcap\limits_{M \in \mu(F)} \Omega(M)$.*

Proof. Let $x \in \mathcal{K}(F)$ and $F \subseteq M$ where $M \in \mu$. Then

$$L = (x)^\circ \vee F \subseteq (x)^\circ \vee M.$$

Suppose $(x)^\circ \subseteq M$, then $M = L$, which is a contradiction. Hence $(x)^\circ \nsubseteq M$. Thus $x \in \Omega(M)$ for all $M \in \mu(F)$. Hence $\mathcal{K}(F) \subseteq \bigcap\limits_{M \in \mu(F)} \Omega(M)$. Conversely, let $x \in \bigcap\limits_{M \in \mu(F)} \Omega(M)$. Then $x \in \Omega(M)$ for all $M \in \mu(F)$. Suppose $(x)^\circ \vee F \neq L$. Then there exists a maximal filter M_0 such that $(x)^\circ \vee F \subseteq M_0$. Hence $(x)^\circ \subseteq M_0$ and $F \subseteq M$. Since $F \subseteq M_0$, by hypothesis, we get $x \in \Omega(M_0)$. Hence $(x)^\circ \nsubseteq M_0$, which is a contradiction. Therefore $(x)^\circ \vee F = L$. Hence $x \in \mathcal{K}(F)$. Therefore $\bigcap\limits_{M \in \mu(F)} \Omega(M) \subseteq \mathcal{K}(F)$. □

From the above theorem, it can be easily observed that $\mathcal{K}(F) \subseteq \Omega(M)$ for every $M \in \mu(F)$. Now, in the following, a set of equivalent conditions is derived for the class of all D-filters of the form $\mathcal{K}(F)$ to become a sublattice to the lattice $\mathcal{F}(L)$ of all filters of L, which leads to a characterization of a quasi-complemented lattice.

Theorem 2.18. *The following assertions are equivalent in lattice L:*

(1) *L is quasi-complemented;*
(2) *for any $M \in \mu$, $\Omega(M)$ is maximal;*
(3) *for any $F, G \in \mathcal{F}(L)$, $F \vee G = L$ implies $\mathcal{K}(F) \vee \mathcal{K}(G) = L$;*
(4) *for any $F, G \in \mathcal{F}(L)$, $\mathcal{K}(F) \vee \mathcal{K}(G) = \mathcal{K}(F \vee G)$;*
(5) *for any two distinct maximal filters M and N, $\Omega(M) \vee \Omega(N) = L$;*
(6) *for any $M \in \mu$, M is the unique member of μ such that $\Omega(M) \subseteq M$.*

Proof. (1) $\Rightarrow$ (2): Assume that L is quasi-complemented. Let M be a maximal filter of L. It is enough to show that $\Omega(M) = M$. Clearly $\Omega(M) \subseteq M$. On the other hand, let $x \in M$. Since L is quasi-complemented, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D(L)$. Hence $y \in (x)^\circ$. If $y \in M$, then $0 = x \wedge y \in M$ which is a contradiction. Hence $y \notin M$ such that $y \in (x)^\circ$. Thus $(x)^\circ \nsubseteq M$. Hence $x \in \Omega(M)$. Therefore $M \subseteq \Omega(M)$.

(2) $\Rightarrow$ (3): Clearly $\Omega(M) = M$ for all $M \in \mu$. Let $F, G \in \mathcal{F}(L)$ be such that $F \vee G = L$. Suppose $\mathcal{K}(F) \vee \mathcal{K}(G) \neq L$. Then there exists a maximal filter M such that $\mathcal{K}(F) \vee \mathcal{K}(G) \subseteq M$. Hence $\mathcal{K}(F) \subseteq M$ and $\mathcal{K}(G) \subseteq M$. Now

$$\begin{aligned}
\mathcal{K}(F) \subseteq M \Rightarrow & \bigcap_{M_i \in \mu(F)} \Omega(M_i) \subseteq M \\
& \Rightarrow \Omega(M_i) \subseteq M \quad \text{for some } M_i \in \mu(F) \text{ (since } M \text{ is prime)} \\
& \Rightarrow M_i \subseteq M \quad \text{by (2)} \\
& \Rightarrow F \subseteq M \quad \text{since } F \subseteq M_i.
\end{aligned}$$

Similarly, we get $G \subseteq M$. Hence $L = F \vee G \subseteq M$, which is a contradiction to the maximality of M. Therefore $\mathcal{K}(F) \vee \mathcal{K}(G) = L$.

(3) $\Rightarrow$ (4): Let $F, G \in \mathcal{F}(L)$. Clearly $\mathcal{K}(F) \vee \mathcal{K}(G) \subseteq \mathcal{K}(F \vee G)$. Conversely, let $x \in \mathcal{K}(F \vee G)$. Then $\{(x)^\circ \vee F\} \vee \{(x)^\circ \vee G\} = (x)^\circ \vee F \vee G = L$. Hence by condition (3), we get $\mathcal{K}((x)^\circ \vee F) \vee \mathcal{K}((x)^\circ \vee G) = L$. Thus $x \in \mathcal{K}((x)^\circ \vee F) \vee \mathcal{K}((x)^\circ \vee G)$. Hence $x = r \wedge s$ for some $r \in \mathcal{K}((x)^\circ \vee F)$

and $s \in \mathcal{K}((x)^{\circ} \vee G)$. Now

$$\begin{aligned} r \in \mathcal{K}((x)^{\circ} \vee F) &\Rightarrow (r)^{\circ} \vee \{(x)^{\circ} \vee F\} = L \\ &\Rightarrow L = \{(r)^{\circ} \vee (x)^{\circ}\} \vee F \subseteq (r \vee x)^{\circ} \vee F \\ &\Rightarrow (r \vee x)^{\circ} \vee F = L \\ &\Rightarrow r \vee x \in \mathcal{K}(F) \end{aligned}$$

Similarly, we get $s \vee x \in \mathcal{K}(G)$. Now, we have the following consequence:

$$\begin{aligned} x &= x \vee x \\ &= (r \wedge s) \vee x \\ &= (r \vee x) \wedge (s \vee x) \end{aligned}$$

where $r \vee x \in \mathcal{K}(F)$ and $s \vee x \in \mathcal{K}(G)$. Hence $x \in \mathcal{K}(F) \vee \mathcal{K}(G)$. Thus $\mathcal{K}(F \vee G) \subseteq \mathcal{K}(F) \vee \mathcal{K}(G)$. Therefore $\mathcal{K}(F) \vee \mathcal{K}(G) = \mathcal{K}(F \vee G)$.

(4) $\Rightarrow$ (5): Let M, N be two distinct maximal filters of L. Choose $x \in M - N$ and $y \in N - M$. Since $x \notin N$, we get $N \vee [x) = L$. Since $y \notin M$, we get $M \vee [y) = L$. Now, we get

$$\begin{aligned} L &= \mathcal{K}(L) \\ &= \mathcal{K}(L \vee L) \\ &= \mathcal{K}\big(\{N \vee [x)\} \vee \{M \vee [y)\}\big) \\ &= \mathcal{K}\big(\{M \vee [x)\} \vee \{N \vee [y)\}\big) \\ &= \mathcal{K}(M \vee N) && \text{since } x \in M \text{ and } y \in N \\ &= \mathcal{K}(M) \vee \mathcal{K}(N) && \text{by condition (4)} \\ &\subseteq \Omega(M) \vee \Omega(N) && \text{by Proposition 2.16(1)} \end{aligned}$$

Therefore $\Omega(M) \vee \Omega(N) = L$.

(5) $\Rightarrow$ (6): Let $M \in \mu$. Suppose $N \in \mu$ such that $N \neq M$ and $\Omega(N) \subseteq M$. Since $\Omega(M) \subseteq M$, by hypothesis, we get $L = \Omega(M) \vee \Omega(N) = M$, which is a contradiction. Hence M is the unique maximal filter such that $\Omega(M) \subseteq M$.

(6) $\Rightarrow$ (1): Let $x \in L$. Suppose $0 \notin [x) \vee (x)^{\circ}$. Then there exist a maximal filter M such that $[x) \vee (x)^{\circ} \subseteq M$. Then $x \in M$ and $(x)^{\circ} \subseteq M$. Hence $x \in M$ and $x \notin \Omega(M)$. Since $x \notin \Omega(M)$, there exists a maximal filter M_0 such that $x \notin M_0$ and $\Omega(M) \subseteq M_0$. By the uniqueness of M, we get $M = M_0$. Hence $x \notin M_0 = M$, which is a contradiction. Thus $0 \in [x) \vee (x)^{\circ}$, which gives $0 = x \wedge a$ for some $a \in (x)^{\circ}$. Hence $x \wedge a = 0$ and $x \vee a \in D(L)$. Therefore L is a quasi-complemented lattice. $\square$

Theorem 2.19. *Following assertions are equivalent in a lattice L:*

(1) *L is quasi-complemented;*
(2) *every D-filter is a $\mathcal{K}$-filter;*
(3) *every prime D-filter is a $\mathcal{K}$-filter;*
(4) *every prime D-filter is minimal.*

Proof. (1) $\Rightarrow$ (2): Assume that L is quasi-complemented. Let F be a D-filter of L. Clearly $\mathcal{K}(F) \subseteq F$. On the other hand, let $x \in F$. Since L is quasi-complemented, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. Suppose $(x)^{\circ} \vee F \neq L$. Then there exists a prime filter P such that $(x)^{\circ} \vee F \subseteq P$. Then $(x)^{\circ} \subseteq P$ and $x \in F \subseteq P$. Suppose $y \in P$. Then $0 = x \wedge y \in P$ which is a contradiction. Hence $y \notin P$. Since $x \vee y \in D$, we get $y \in (x)^{\circ} \subseteq P$ yields a contradiction. Thus $(x)^{\circ} \vee F = L$ which gives that $x \in \mathcal{K}(F)$. Hence $F \subseteq \mathcal{K}(F)$. Therefore F is $\mathcal{K}$-filter of L.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (4): Assume that every prime D-filter is a $\mathcal{K}$-filter. Let P be a prime D-filter of L. Since P is proper, there exists $c \in L$ such that $c \notin P$. By condition (3), P is a $\mathcal{K}$-filter of L. Hence $\mathcal{K}(P) = P$. Let $x \in P = \mathcal{K}(P)$. Then $(x)^{\circ} \vee P = L$ and thus $c \in (x)^{\circ} \vee P$. Then $c = a \wedge b$ for some $a \in (x)^{\circ}$ and $b \in P$. Since $a \in (x)^{\circ}$, we get $x \vee a \in D(L)$. Suppose $a \in P$. Since P is prime and $b \in P$, we get $c = a \wedge b \in P$ which is a contradiction. Thus $a \notin P$. Hence $x \vee a \in D(L)$ for some $a \notin P$. Therefore P is minimal.

(4) $\Rightarrow$ (1): By Theorem 2.2, it follows. □

Since every Boolean algebra contains a unique dense element precisely 1, it is clear that every filter of a Boolean algebra is a D-filter. Further, it can be easily seen that every Boolean algebra is quasi-complemented. Thus, we have the following:

Theorem 2.20. *Following assertions are equivalent in a lattice L:*

(1) *L is a Boolean algebra;*
(2) *every filter is a $\mathcal{K}$-filter;*
(3) *every prime filter is a $\mathcal{K}$-filter;*
(4) *every prime filter is minimal.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are straightforward.

(4) $\Rightarrow$ (1): Assume that every prime filter of L is minimal. Let $x \in L$. Suppose $0 \notin [x) \vee (x)^{+}$. Then there exists a prime filter P such that $[x) \vee (x)^{+} \subseteq P$. Hence $x \in P$ and $(x)^{+} \subseteq P$. Since P is minimal and $(x)^{+} \subseteq P$, we get $x \notin P$ which is a contraction. Hence $0 \in [x) \vee (x)^{+}$. Then

there exist $a \in (x)^+$ such that $a \wedge x = 0$. Since $a \in (x)^+$, we get $x \vee a = 1$. Hence a is the complement of x. Therefore L is Boolean. □

Acknowledgments

The author would like to thank the referee for his valuable suggestions and comments which improved the presentation of this article.

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Journal of Algebraic Systems

$\mathcal{K}$-FILTERS OF DISTRIBUTIVE LATTICES

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$\mathcal{K}$-فیلترهای مشبکه‌های توزیع‌پذیر

ام. سامباسیوارائو

گروه ریاضی، کالج مهندسی MVGR، ویزیاناگارام، آندراپرادش، هند

در این مقاله، مفهوم $\mathcal{K}$-فیلترها در مشبکه‌های توزیع‌پذیر معرفی شده و برخی از ویژگی‌های این دسته از فیلترها مورد مطالعه گرفته است. برخی شرایط لازم و کافی برای اینکه هر π-فیلتر یک مشبکه توزیع‌پذیر به یک $\mathcal{K}$-فیلتر تبدیل شود، ارائه شده است. همچنین، برخی شرایط معادل برای اینکه هر D-فیلتر یک مشبکه توزیع‌پذیر به $\mathcal{K}$-فیلتر تبدیل شود، بیان شده است. به علاوه، مشبکه‌های شبه-کامل‌شده با کمک $\mathcal{K}$-فیلترها مشخصه‌سازی شده‌اند.

کلمات کلیدی: D-فیلتر، فیلتر منظم، π-فیلتر، ω-فیلتر، $\mathcal{K}$-فیلتر، مشبکه شبه-کامل‌شده.