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STRUCTURED CONDITION PSEUDOSPECTRA AND STRUCTURED ESSENTIAL CONDITION PSEUDOSPECTRA OF BOUNDED LINEAR OPERATORS ON ULTRAMETRIC BANACH SPACES

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Abstract. In this paper, we introduce and study the structured condition pseudospectra and the structured essential condition pseudospectra of bounded linear operators on ultrametric Banach spaces. We establish a characterization of the structured condition pseudospectrum of continuous linear operators and we give a relationship between the structured condition pseudospectrum and the structured pseudospectrum of continuous linear operators on ultrametric Banach spaces. Many characterizations of the structured essential condition pseudospectrum of bounded linear operators and examples are given.

1. Introduction and Preliminaries

In the classical analysis, Trefethen [16] introduced the pseudospectra of matrices and bounded linear operators. Moreover, Davies [8] introduced the structured pseudospectrum of a closed linear operator S on a complex Banach space $\mathcal{E}$ as follows:

$$\sigma_\varepsilon(S,B,C)=\bigcup_{D\in\mathcal{B}(\mathcal{F},\mathcal{G}):\|D\|<\varepsilon}\sigma(S+CDB),$$

where $\mathcal{F}$ and $\mathcal{G}$ are complex Banach spaces, $B\in\mathcal{B}(\mathcal{E},\mathcal{F})$ and $C\in\mathcal{B}(\mathcal{G},\mathcal{E})$. He gave a characterization of the structured pseudospectrum of the closed linear operator S. For more details, we refer to [8].

In ultrametric operator theory, the authors [3] extended and studied the pseudospectra of linear operators on ultrametric Banach spaces. They characterized the pseudospectra of linear operators and the essential pseudospectra of closed linear operators. Furthermore, they established a relationship between the essential pseudospectra and the essential pseudospectra of closed linear operators perturbed by completely continuous operators on ultrametric Hilbert spaces. In [4], Ammar, Bouchekoua and Lazrag introduced and

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studied the condition pseudospectra of bounded linear operators on ultrametric Banach spaces. They gave a relationship between the condition pseudospectra and the pseudospectra and they obtained some properties of the essential condition pseudospectrum of bounded linear operators on ultrametric Banach spaces. Recently, EL Amrani, Ettayb and Blali [11] studied the pseudospectra and the condition pseudospectra of ultrametric matrices and the pseudospectra of ultrametric matrix pencils. They proved some results about them and they gave some illustrative examples. Furthermore, the trace pseudospectra, the determinant pseudospectra of ultrametric matrix pencils and the condition pseudospectra of ultrametric operator pencils were studied by several authors. For more details, we refer to [6], [7], [11] and [14].

The eigenvalue problem is one of interesting problems in ultrametric operator theory. It played an important role in many parts of ultrametric applied mathematics and physics including matrix theory, ultrametric pseudo-differential equations, control theory and ultrametric quantum mechanics. For more details, we refer to [1], [2], [3] and [4]. This work is motivated by many studies of ultrametric spectral theory and perturbation theory of bounded linear operators. For more details, we refer to [3, 4, 6, 7, 11, 14].

Throughout this paper, $\mathbb{K}$ is a complete ultrametric field with non-trivial valuation $|\cdot|$, $\mathcal{E}$ is an ultrametric Banach space over $\mathbb{K}$, $\mathcal{B}(\mathcal{E})$ denotes the collection of all continuous linear operators on $\mathcal{E}$, $\mathcal{E}^* = \mathcal{B}(\mathcal{E}, \mathbb{K})$ is the dual space of $\mathcal{E}$ and $\mathbb{Q}_p$ is the field of p-adic numbers. For more details, we refer to [9, 17]. For $S \in \mathcal{B}(\mathcal{E})$, $N(S), R(S), \rho(S), \sigma(S)$ and $\sigma_e(S)$ are the kernel, the range, the resolvent set, the spectrum and the essential spectrum of S respectively. For more details, see [4, 9]. We begin with the following preliminaries.

Definition 1.1. [9] Let $\mathcal{E}$ be a vector space over $\mathbb{K}$. A function $\|\cdot\| : \mathcal{E} \to \mathbb{R}_+$ is called an ultrametric norm if:

(i) For each $u \in \mathcal{E}$, $\|u\| = 0$ if and only if $u = 0$;
(ii) For all $u \in \mathcal{E}$ and $\lambda \in \mathbb{K}$, $\|\lambda u\| = |\lambda|\|u\|$;
(iii) For any $u, v \in \mathcal{E}$, $\|u + v\| \leq \max(\|u\|, \|v\|)$.

Definition 1.2. [9] An ultrametric Banach space is a complete ultrametric normed space.

Theorem 1.3. [17] *Assume that* $\mathbb{K}$ *is spherically complete. Let* $\mathcal{E}$ *be an ultrametric Banach space over* $\mathbb{K}$*. For all* $u \in \mathcal{E}\backslash\{0\}$*, there is* $u^* \in \mathcal{E}^*$ *such that* $u^*(u) = 1$ *and* $\|u^*\| = \|u\|^{-1}$.

Definition 1.4. [9] Let $\omega = (\omega_i)_i$ be a sequence of $\mathbb{K}\backslash\{0\}$. We define $\mathcal{E}_\omega$ by

$$\mathcal{E}_\omega = \{u = (u_i)_i : \forall i \in \mathbb{N},\ u_i \in \mathbb{K} \text{ and } \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|u_i| = 0\}.$$

On $\mathcal{E}_\omega$, we define

$$\forall u \in \mathcal{E}_\omega : u = (u_i)_i,\ \|u\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|u_i|).$$

Then $(\mathcal{E}_\omega, \|\cdot\|)$ is an ultrametric Banach space.

Remark 1.5. [9] The orthogonal basis $\{e_i,\ i \in \mathbb{N}\}$ is called the canonical basis of $\mathcal{E}_\omega$ where $e_i = (\delta_{i,j})_{j\in\mathbb{N}}$ and $\delta_{i,j}$ is the Kronecker symbol. For each $i \in \mathbb{N}$, $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

Definition 1.6. [15] Let $S \in \mathcal{B}(\mathcal{E})$, S is called an upper semi-Fredholm operator if

$$\alpha(S) = \dim N(S) \text{ is finite and R(S) is closed.}$$

The collection of all upper semi-Fredholm operators on $\mathcal{E}$ is denoted by $\Phi_+(\mathcal{E})$.

Definition 1.7. [15] Let $S \in \mathcal{B}(\mathcal{E})$, S is said to be a lower semi-Fredholm operator if $\beta(S) = \dim(\mathcal{E}/R(S))$ is finite.

The collection of all lower semi-Fredholm operators on $\mathcal{E}$ is denoted by $\Phi_-(\mathcal{E})$.

The collection of all bounded Fredholm operators on $\mathcal{E}$ is

$$\Phi(\mathcal{E}) = \Phi_+(\mathcal{E}) \cap \Phi_-(\mathcal{E}).$$

Let $S \in \Phi(\mathcal{E})$, the index $ind(S)$ of S is defined by $ind(S) = \alpha(S) - \beta(S)$. For more details on bounded Fredholm operators, see [5], [10], [15] and [17].

Definition 1.8. [17] Let $S \in \mathcal{B}(\mathcal{E})$, S is said to be an operator of finite rank if $\dim R(S)$ is finite.

The set of all finite rank operators on $\mathcal{E}$ will be denoted by $\mathcal{F}_0(\mathcal{E})$.

Definition 1.9. [9] Let $\mathcal{E}$ be an ultrametric Banach space and let $S \in \mathcal{B}(\mathcal{E})$, S is called completely continuous if, there is a sequence of finite rank linear operators $(S_n)_{n\in\mathbb{N}}$ such that $\|S_n - S\| \to 0$ as $n \to \infty$.

$\mathcal{C}_c(\mathcal{E})$ is the set of all completely continuous linear operators on $\mathcal{E}$.

Lemma 1.10. [10] *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(\mathcal{E})$ and $C \in \mathcal{C}_c(\mathcal{E})$, hence $S + C \in \Phi(\mathcal{E})$.*

The following lemma showed that the index of a Fredholm operator between ultrametric Banach spaces is preserved under completely continuous perturbations.

Lemma 1.11. [4] *Suppose that* $\mathbb{K}$ *is spherically complete. If* $S \in \Phi(\mathcal{E})$, *hence for each* $C \in \mathcal{C}_c(\mathcal{E})$, $S + C \in \Phi(\mathcal{E})$ *and* $ind(S + C) = ind(S)$.

Theorem 1.12. [4] *Suppose that* $\mathbb{K}$ *is spherically complete. Let* $S \in \mathcal{B}(\mathcal{E})$. *Then*

$$\sigma_e(S) = \bigcap_{K \in \mathcal{C}_c(\mathcal{E})} \sigma(S + K).$$

As the classical setting, we have.

Proposition 1.13. [8] *Let* $\mathcal{E}$ *be an ultrametric Banach space over* $\mathbb{K}$. *If* $S, B \in \mathcal{B}(\mathcal{E})$, *then* $-1 \notin \sigma(SB)$ *if, and only if,* $-1 \notin \sigma(BS)$.

Lemma 1.14. [9] *Let* $\mathcal{E}$ *be an ultrametric Banach space over* $\mathbb{K}$. *Let* $S \in \mathcal{B}(\mathcal{E})$ *such that* $\|S\| < 1$, *then* $(I - S)^{-1} \in \mathcal{B}(\mathcal{E})$ *and* $\|(I - S)^{-1}\| \leq 1$.

Let $\mathcal{M}_n(\mathbb{K})$ be the algebra of all $n \times n$ matrices with entries in $\mathbb{K}$. We have the following definitions.

Definition 1.15. [11] Let $S \in \mathcal{M}_n(\mathbb{K})$. The spectrum $\sigma(S)$ of S is

$$\sigma(S) = \{\lambda \in \mathbb{K} : S - \lambda I \text{ is not invertible}\}.$$

The resolvent set $\rho(S)$ of S is $\mathbb{K} \backslash \sigma(S)$.

Definition 1.16. [11] Let $S \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S)$ of S is defined by

$$\sigma_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \ \|(S - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

By convention $\|(S - \lambda I)^{-1}\| = \infty$ if $\lambda \in \sigma(S)$.

Proposition 1.17. [11] *Let* $S \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$. *Hence*

(i) $\sigma(S) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(S)$;

(ii) *For each* ε_1 *and* ε_2 *with* $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(S) \subset \sigma_{\varepsilon_1}(S) \subset \sigma_{\varepsilon_2}(S)$.

Theorem 1.18. [11] *Let* $\mathcal{E}$ *be a finite-dimensional ultrametric Banach space over* $\mathbb{Q}_p$ *such that* $\|\mathcal{E}\| \subseteq |\mathbb{Q}_p|$, *let* $S \in \mathcal{B}(\mathcal{E})$ *and* $\varepsilon > 0$. *Then*

$$\sigma_\varepsilon(S) = \bigcup_{C \in \mathcal{B}(\mathcal{E}) : \|C\| < \varepsilon} \sigma(S + C).$$

The following definition is introduced.

Definition 1.19. [11] Let $S \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The condition pseudospectrum $\Lambda_\varepsilon(S)$ of S is given by

$$\Lambda_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \ \|(S-\lambda I)\|\|(S-\lambda I)^{-1}\| > \varepsilon^{-1}\},$$

by convention $\|(S-\lambda I)\|\|(S-\lambda I)^{-1}\| = \infty$ if and only if $\lambda \in \sigma(S)$. The condition pseudoresolvent of S is $\mathbb{K}\backslash\Lambda_\varepsilon(S)$.

2. Main Results

We start with the following definition.

Definition 2.1. Let $S, B, C \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The structured pseudospectrum $\sigma_\varepsilon(S, B, C)$ of S is defined by

$$\sigma_\varepsilon(S, B, C) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \ \|B(S-\lambda I)^{-1}C\| > \varepsilon^{-1}\}.$$

The structured pseudoresolvent $\rho_\varepsilon(S, B, C)$ of S is defined by

$$\rho_\varepsilon(S, B, C) = \rho(S) \cap \{\lambda \in \mathbb{K} : \|B(S-\lambda I)^{-1}C\| \leq \varepsilon^{-1}\},$$

by convention $\|B(S-\lambda I)^{-1}C\| = \infty$ if, and only if, $\lambda \in \sigma(S)$.

We collect some properties of the structured pseudospectra of ultrametric matrices.

Proposition 2.2. *Let $S, B, C \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Then*

(i) $\sigma(S) = \bigcap_{\varepsilon>0} \sigma_\varepsilon(S, B, C)$;

(ii) *For all ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$,*

$$\sigma(S) \subset \sigma_{\varepsilon_1}(S, B, C) \subset \sigma_{\varepsilon_2}(S, B, C).$$

Proof. The proof is similar to the proof of Proposition 1.17. □

Theorem 2.3. *Let $S, B, C \in \mathcal{B}(\mathbb{Q}_p^n)$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon(S, B, C) = \bigcup_{D \in \mathcal{B}(\mathbb{Q}_p^n) : \|D\| < \varepsilon} \sigma(S + CDB).$$

Proof. The proof is similar to the proof of Theorem 1.18. □

Example 2.4. Let $\varepsilon > 0, \lambda_1, \lambda_2 \in \mathbb{Q}_p\backslash\{0\}$ and let

$$S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then

$$\begin{aligned}\sigma_\varepsilon(S,B,C) &= \sigma(S)\cup\{\lambda\in\mathbb{Q}_p : \|B(S-\lambda I)^{-1}C\| > \frac{1}{\varepsilon}\}\\ &= \{\lambda_1,\lambda_2\}\cup\{\lambda\in\mathbb{Q}_p : |\lambda-\lambda_1|<\varepsilon\}.\end{aligned}$$

Example 2.5. Let $\mathbb{K}=\mathbb{Q}_p, \varepsilon>0$ and let

$$S=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0\\1&1\end{pmatrix} \text{ and } C=I=\begin{pmatrix}1&0\\0&1\end{pmatrix}\in\mathcal{M}_2(\mathbb{Q}_p).$$

Hence

$$\begin{aligned}\sigma_\varepsilon(S,B,C) &= \sigma(S)\cup\{\lambda\in\mathbb{Q}_p : \|B(S-\lambda I)^{-1}C\| > \frac{1}{\varepsilon}\}\\ &= \{0,1\}\cup\{\lambda\in\mathbb{Q}_p : \max\{\frac{1}{|1-\lambda|},\frac{1}{|\lambda|}\} > \frac{1}{\varepsilon}\}.\end{aligned}$$

Example 2.6. Let $\mathbb{K}=\mathbb{Q}_p, \varepsilon>0$ and let

$$S=\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad B=\begin{pmatrix}1&0\\0&0\end{pmatrix} \text{ and } C=\begin{pmatrix}0&1\\0&0\end{pmatrix}\in\mathcal{M}_2(\mathbb{Q}_p).$$

Consequently

$$\begin{aligned}\sigma_\varepsilon(S,B,C) &= \sigma(S)\cup\{\lambda\in\mathbb{Q}_p : \|B(S-\lambda I)^{-1}C\| > \frac{1}{\varepsilon}\}\\ &= \{1\}\cup\{\lambda\in\mathbb{Q}_p : |\lambda-1|<\varepsilon\}.\end{aligned}$$

We introduce the following definition.

Definition 2.7. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$. Let $S,B,C\in\mathcal{B}(\mathcal{E})$ such that $0\in\rho(B)\cap\rho(C)$ and $\varepsilon>0$, the structured condition pseudospectrum $\Lambda_\varepsilon(S,B,C)$ of S is defined by

$$\Lambda_\varepsilon(S,B,C)=\sigma(S)\cup\{\lambda\in\mathbb{K} : \|C^{-1}(S-\lambda I)B^{-1}\|\|B(S-\lambda I)^{-1}C\| > \frac{1}{\varepsilon}\}.$$

The structured condition pseudoresolvent of S is given by

$$\rho(S)\cap\{\lambda\in\mathbb{K} : \|C^{-1}(S-\lambda I)B^{-1}\|\|B(S-\lambda I)^{-1}C\| \le \frac{1}{\varepsilon}\}.$$

By convention $\|C^{-1}(S-\lambda I)B^{-1}\|\|B(S-\lambda I)^{-1}C\|=\infty$ if $\lambda\in\sigma(S)$.

From Definition 2.7, we conclude the following remark.

Remark 2.8. If $C=B=I$, hence $\Lambda_\varepsilon(S,I,I)=\Lambda_\varepsilon(S)$ (the condition pseudospectrum of S).

By Definition 2.7, it follows that the structured condition pseudospectra associated with various ε are nested sets and the intersection of all the structured condition pseudospectra is the spectrum.

Proposition 2.9. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $S, B, C \in \mathcal{B}(\mathcal{E})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$, then*

(i) $\sigma(S) = \bigcap_{\varepsilon>0} \Lambda_\varepsilon(S, B, C)$;

(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, hence $\sigma(S) \subset \Lambda_{\varepsilon_1}(S, B, C) \subset \Lambda_{\varepsilon_2}(S, B, C)$.*

Proof. (i) From Definition 2.7, for each $\varepsilon > 0$, $\sigma(S) \subset \Lambda_\varepsilon(S, B, C)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Lambda_\varepsilon(S, B, C)$, hence for each $\varepsilon > 0$, $\lambda \in \Lambda_\varepsilon(S, B, C)$. If $\lambda \notin \sigma(S)$, hence $\lambda \in \{\lambda \in \mathbb{K} : \|C^{-1}(S - \lambda I)B^{-1}\|\|B(S - \lambda I)^{-1}C\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \to 0^+$, we have $\|C^{-1}(S - \lambda I)B^{-1}\|\|B(S - \lambda I)^{-1}C\| = \infty$. Then $\lambda \in \sigma(S)$.

(ii) If $\lambda \in \Lambda_{\varepsilon_1}(S, B, C)$, hence

$\|C^{-1}(S - \lambda I)B^{-1}\|\|B(S - \lambda I)^{-1}C\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$. Thus $\lambda \in \Lambda_{\varepsilon_2}(S, B, C)$.

□

The next lemma establishes a characterization of the structured condition pseudospectra of continuous linear operators on $\mathcal{E}$.

Lemma 2.10. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Hence $\lambda \in \Lambda_\varepsilon(S, B, C) \setminus \sigma(S)$ if, and only if, there is $x \in \mathcal{E}\setminus\{0\}$ such that*

$$\|C^{-1}(S - \lambda I)B^{-1}x\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|\|x\|.$$

Proof. If $\lambda \in \Lambda_\varepsilon(S, B, C) \setminus \sigma(S)$, then

$$\|C^{-1}(S - \lambda I)B^{-1}\|\|B(S - \lambda I)^{-1}C\| > \varepsilon^{-1}.$$

Thus $\|B(S - \lambda I)^{-1}C\| > \frac{1}{\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|}$. Hence

$$\sup_{y\in\mathcal{E}\setminus\{0\}} \frac{\|B(S - \lambda I)^{-1}Cy\|}{\|y\|} > \frac{1}{\varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|}.$$

Then, there is $y \in \mathcal{E}\setminus\{0\}$ with $\|B(S - \lambda I)^{-1}Cy\| > \frac{\|y\|}{\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|}$. Setting $x = B(S - \lambda I)^{-1}Cy$, hence $y = C^{-1}(S - \lambda I)B^{-1}x$. Consequently

$$\|x\| > \frac{\|C^{-1}(S - \lambda I)B^{-1}x\|}{\varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|}.$$

Hence,

$$\|C^{-1}(S-\lambda I)B^{-1}x\| < \varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|\|x\|.$$

Conversely, assume that there is $x \in \mathcal{E}\backslash\{0\}$ such that

$$\|C^{-1}(S-\lambda I)B^{-1}x\| < \varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|\|x\|. \tag{2.1}$$

If $\lambda \notin \sigma(S)$ and $x = B(S-\lambda I)^{-1}Cy$, then

$$\|x\| \leq \|B(S-\lambda I)^{-1}C\|\|y\|.$$

From (2.1) and $y = C^{-1}(S-\lambda I)B^{-1}x$, we have

$$\|x\| < \varepsilon\|B(S-\lambda I)^{-1}C\|\|C^{-1}(S-\lambda I)B^{-1}\|\|x\|.$$

Then

$$\|B(S-\lambda I)^{-1}C\|\|C^{-1}(S-\lambda I)B^{-1}\| > \frac{1}{\varepsilon}.$$

Consequently, $\lambda \in \Lambda_\varepsilon(S,B,C)\backslash\sigma(S)$. □

The following theorem gives a relationship between the structured condition pseudospectra of a continuous linear operator and the structured condition pseudospectra of its inverse.

Theorem 2.11. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$ and let $\varepsilon > 0$. Let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(S)\cap\rho(B)\cap\rho(C)$ and $SC = CS$. Set $k = \|S^{-1}\|\|S\|$. Then the following statements hold.*

(i) *If $\lambda \in \Lambda_\varepsilon(S^{-1},B,C)\backslash\{0\}$, then $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(S,B,C)\backslash\{0\}$;*
(ii) *If $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(S,B,C)\backslash\{0\}$, then $\lambda \in \Lambda_{\varepsilon k^2}(S^{-1},B,C)\backslash\{0\}$.*

Proof. (i) If $\lambda \in \Lambda_\varepsilon(S^{-1},B,C)\backslash\{0\}$, thus

$$\begin{aligned}\frac{1}{\varepsilon} &< \|C^{-1}(S^{-1}-\lambda I)B^{-1}\|\|B(S^{-1}-\lambda I)^{-1}C\| \\ &= \|\lambda C^{-1}S^{-1}\Big(\frac{I}{\lambda}-S\Big)B^{-1}\| \times \|\lambda^{-1}B\Big(\frac{I}{\lambda}-S\Big)^{-1}SC\| \\ &\leq \|S^{-1}\|\|S\|\|C^{-1}\Big(\frac{I}{\lambda}-S\Big)B^{-1}\| \times \|B\Big(\frac{I}{\lambda}-S\Big)^{-1}C\|.\end{aligned}$$

Hence $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(S,B,C)\backslash\{0\}$.

(ii) If $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(S,B,C)\backslash\{0\}$, hence

$$\begin{aligned}\frac{1}{\varepsilon k} &< \|C^{-1}(S-\lambda^{-1}I)B^{-1}\|\|B(S-\lambda^{-1}I)^{-1}C\| \\ &= \|\lambda^{-1}C^{-1}S\Big(\lambda I-S^{-1}\Big)B^{-1}\| \times \|\lambda B\Big(\lambda I-S^{-1}\Big)^{-1}S^{-1}C\|\end{aligned}$$

$$\leq \ \|S^{-1}\|\|S\|\|C^{-1}(\lambda I - S^{-1})B^{-1}\| \times \|B\Big(\lambda I - S^{-1}\Big)^{-1}C\|.$$

Then

$$\frac{1}{\varepsilon k^2} < \|C^{-1}(\lambda I - S^{-1})B^{-1}\|\|B\Big(\lambda I - S^{-1}\Big)^{-1}C\|.$$

Consequently, $\lambda \in \Lambda_{\varepsilon k^2}(S^{-1}, B, C)\backslash\{0\}$. □

Theorem 2.12. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. If there is $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$ and $\lambda \in \sigma(S + CDB)$. Hence $\lambda \in \Lambda_\varepsilon(S, B, C)$.*

Proof. Assume that there exists $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$ and $\lambda \in \sigma(S + CDB)$. Let $\lambda \notin \Lambda_\varepsilon(S, B, C)$, thus $\lambda \in \rho(S)$ and

$$\|C^{-1}(S - \lambda I)B^{-1}\|\|B(S - \lambda I)^{-1}C\| \leq \varepsilon^{-1}. \tag{2.2}$$

Consider F defined on $\mathcal{E}$ by

$$F = \sum_{n=0}^{\infty}(S - \lambda I)^{-1}C\Big(- DB(S - \lambda I)^{-1}C\Big)^n C^{-1}. \tag{2.3}$$

It follows from (2.2) and Lemma 1.14 that

$$F = (S - \lambda I)^{-1}C\Big(C + CDB(S - \lambda I)^{-1}C\Big)^{-1}.$$

Then $S + CDB - \lambda I$ is invertible and $F = (S + CDB - \lambda I)^{-1} \in \mathcal{B}(\mathcal{E})$ which is a contradiction. Thus $\lambda \in \Lambda_\varepsilon(S, B, C)$. □

We put $\mathcal{D}_\varepsilon(\mathcal{E}) = \{D \in \mathcal{B}(\mathcal{E}) : \|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|\}$.

Theorem 2.13. *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ with $\|\mathcal{E}\| \subseteq |\mathbb{K}|$, let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Hence*

$$\Lambda_\varepsilon(S, B, C) = \bigcup_{D \in \mathcal{D}_\varepsilon(\mathcal{E})} \sigma(S + CDB).$$

Proof. From Theorem 2.12, we get $\displaystyle\bigcup_{D \in \mathcal{D}_\varepsilon(\mathcal{E})} \sigma(S + CDB) \subseteq \Lambda_\varepsilon(S, B, C)$. Conversely, suppose that $\lambda \in \Lambda_\varepsilon(S, B, C)$. If $\lambda \in \sigma(S)$, we may set $D = 0$. If $\lambda \in \Lambda_\varepsilon(S, B, C)$ and $\lambda \notin \sigma(S)$. From Lemma 2.10 and $\|\mathcal{E}\| \subseteq |\mathbb{K}|$, there exists $x \in \mathcal{E}$ with $\|x\| = 1$ and $\|C^{-1}(S - \lambda I)B^{-1}x\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$. By Theorem 1.3, there is $\varphi \in \mathcal{E}^*$ with $\varphi(x) = 1$ and $\|\varphi\| = \|x\|^{-1} = 1$. Consider D on $\mathcal{E}$ defined by for each $y \in \mathcal{E}$, $Dy = -\varphi(y)C^{-1}(S - \lambda I)B^{-1}x$.

Then $\|D\| < \varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|$. For $x \in \mathcal{E}\backslash\{0\}$, $(S-\lambda I)B^{-1}x + CDx = 0$. Set $z = B^{-1}x \in \mathcal{E}\backslash\{0\}$, we have $(S+CDB-\lambda I)z = 0$, hence $S+CDB-\lambda I$ is not injective, then $S+CDB-\lambda I$ is not invertible. Hence,

$$\lambda \in \bigcup_{D\in\mathcal{D}_\varepsilon(\mathcal{E})} \sigma(S+CDB).$$

□

Example 2.14. Let $\varepsilon > 0, a, b \in \mathbb{Q}_p$ and let

$$S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p),$$

then $\sigma(S) = \{a, b\}$ and $\|B(S-\lambda I)^{-1}C\| = \max\left\{\frac{1}{|a-\lambda|}, \frac{|2|}{|b-\lambda|}\right\}$ and

$$\|C^{-1}(S-\lambda I)B^{-1}\| = \max\left\{|a-\lambda|, \frac{|b-\lambda|}{|2|}\right\}.$$

Thus

$$\Lambda_\varepsilon(S,B,C) =$$
$$\{a,b\} \cup \{\lambda \in \mathbb{Q}_p : \frac{|2(a-\lambda)|}{|b-\lambda|} > \frac{1}{\varepsilon}\} \cup \{\lambda \in \mathbb{Q}_p : \frac{|b-\lambda|}{|2(a-\lambda)|} > \frac{1}{\varepsilon}\}.$$

The following proposition gives a relationship between the structured condition pseudospectra and the structured pseudospectra of continuous linear operators on $\mathcal{E}$.

Proposition 2.15. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}, \varepsilon > 0$ and let $S, B, C \in \mathcal{B}(\mathcal{E})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\|C^{-1}(S-\lambda I)B^{-1}\| \neq 0$. Hence*

(i) $\lambda \in \Lambda_\varepsilon(S,B,C)$ *if and only if* $\lambda \in \sigma_{\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|}(S,B,C)$;
(ii) $\lambda \in \sigma_\varepsilon(S,B,C)$ *if and only if* $\lambda \in \Lambda_{\frac{\varepsilon}{\|C^{-1}(S-\lambda I)B^{-1}\|}}(S,B,C)$.

Proof. (i) If $\lambda \in \Lambda_\varepsilon(S,B,C)$, hence $\lambda \in \sigma(S)$ and

$$\|C^{-1}(S-\lambda I)B^{-1}\|\|B(S-\lambda I)^{-1}C\| > \varepsilon^{-1}.$$

Then $\lambda \in \sigma(S)$ and $\|B(S-\lambda I)^{-1}C\| > \frac{1}{\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|}$. Thus, $\lambda \in \sigma_{\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|}(S,B,C)$. The converse is similar.

(ii) If $\lambda \in \sigma_\varepsilon(S,B,C)$, hence $\lambda \in \sigma(S)$ and $\|B(S-\lambda I)^{-1}C\| > \varepsilon^{-1}$. Thus

$$\lambda \in \sigma(S) \text{ and } \|C^{-1}(S-\lambda I)B^{-1}\|\|B(S-\lambda I)^{-1}C\| > \frac{\|C^{-1}(S-\lambda I)B^{-1}\|}{\varepsilon}.$$

Then, $\lambda \in \Lambda_{\frac{\varepsilon}{\|C^{-1}(S-\lambda I)B^{-1}\|}}(S,B,C)$. The converse is similar. □

Theorem 2.16. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $S,B,C,U \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$ such that $0 \in \rho(B)\cap\rho(C)\cap\rho(U)$, $UC = CU$ and $BU = UB$. Set $V = U^{-1}SU$, then*

$$\Lambda_{\frac{\varepsilon}{k^2}}(S,B,C) \subseteq \Lambda_\varepsilon(V,B,C) \subseteq \Lambda_{k^2\varepsilon}(S,B,C),$$

in which $k = \|U\|\|U^{-1}\|$.

Proof. If $\lambda \in \Lambda_{\frac{\varepsilon}{k^2}}(S,B,C)$, hence $\lambda \in \sigma(S)\Big(=\sigma(V)\Big)$ and

$$\begin{aligned}\frac{k^2}{\varepsilon} &< \|C^{-1}(S-\lambda I)B^{-1}\|\|B(S-\lambda I)^{-1}C\| \\ &= \|C^{-1}U\Big(V-\lambda I\Big)U^{-1}B^{-1}\| \times \|BU\Big(V-\lambda I\Big)^{-1}U^{-1}C\| \\ &\le \Big(\|U\|\|U^{-1}\|\Big)^2\|C^{-1}\Big(V-\lambda I\Big)B^{-1}\| \times \|B\Big(V-\lambda I\Big)^{-1}C\| \\ &\le k^2\|C^{-1}\Big(V-\lambda I\Big)B^{-1}\|\|B\Big(V-\lambda I\Big)^{-1}C\|.\end{aligned}$$

Or $k^2 > 0$, hence $\lambda \in \Lambda_\varepsilon(V,B,C)$. Then $\Lambda_{\frac{\varepsilon}{k^2}}(S,B,C) \subseteq \Lambda_\varepsilon(V,B,C)$. Let $\lambda \in \Lambda_\varepsilon(V,B,C)$. Hence

$$\begin{aligned}\frac{1}{\varepsilon} &< \|C^{-1}(V-\lambda I)B^{-1}\|\|B(V-\lambda I)^{-1}C\| \\ &= \|C^{-1}U^{-1}\Big(S-\lambda I\Big)UB^{-1}\| \times \|BU^{-1}\Big(S-\lambda I\Big)^{-1}UC\| \\ &\le \Big(\|U\|\|U^{-1}\|\Big)^2\|C^{-1}\Big(S-\lambda I\Big)B^{-1}\| \times \|B\Big(S-\lambda I\Big)^{-1}C\| \\ &\le k^2\|C^{-1}\Big(S-\lambda I\Big)B^{-1}\|\|B\Big(S-\lambda I\Big)^{-1}C\|.\end{aligned}$$

Then $\lambda \in \Lambda_{k^2\varepsilon}(S,B,C)$. Thus $\Lambda_\varepsilon(V,B,C) \subseteq \Lambda_{k^2\varepsilon}(S,B,C)$. □

We have the following example.

Example 2.17. Let $S,B,C \in \mathcal{B}(\mathcal{E}_\omega)$ be diagonal operators with $0 \in \rho(B)\cap\rho(C)$ defined by

$$\text{for all } i \in \mathbb{N}, Se_i = a_ie_i, Be_i = b_ie_i \text{ and } Ce_i = c_ie_i$$

where $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}, (c_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ such that $\sup_{i\in\mathbb{N}}|a_i|, \sup_{i\in\mathbb{N}}|b_i|$ and $\sup_{i\in\mathbb{N}}|c_i|$ are finite. One can see that $\sigma(S) = \overline{\{a_i : i \in \mathbb{N}\}}$. For each $\lambda \in \rho(S)$,

$$\|B(S-\lambda I)^{-1}C\| = \sup_{i\in\mathbb{N}} \frac{\|B(S-\lambda I)^{-1}Ce_i\|}{\|e_i\|} = \sup_{i\in\mathbb{N}} \left|\frac{b_i c_i}{a_i - \lambda}\right|$$

and $\|C^{-1}(S-\lambda I)B^{-1}\| = \sup\limits_{i\in\mathbb{N}} |c_i^{-1}(a_i-\lambda)b_i^{-1}|$. Consequently,

$$\Lambda_\varepsilon(S,B,C) = \\ \overline{\{a_i :\ i \in \mathbb{N}\}} \cup \left\{\lambda \in \rho(S) : \sup_{i\in\mathbb{N}} |c_i^{-1}(a_i-\lambda)b_i^{-1}| \sup_{i\in\mathbb{N}} \left|\frac{b_i c_i}{a_i-\lambda}\right| > \frac{1}{\varepsilon}\right\}.$$

We introduce the concept of structured essential condition pseudospectra of continuous linear operators on ultrametric Banach spaces.

Definition 2.18. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. The structured essential condition pseudospectrum $\Lambda_{e,\varepsilon}(S,B,C)$ of S is defined by

$$\mathbb{K}\backslash \Big\{\lambda \in \mathbb{K} : S + CDB - \lambda I \in \Phi_0(\mathcal{E}) \text{ for each } D \in \mathcal{B}(\mathcal{E}), \\ \|D\| < \varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|\Big\},$$

where $\Phi_0(\mathcal{E})$ is the set of continuous Fredholm operators on $\mathcal{E}$ of index 0.

We have the following theorem.

Theorem 2.19. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$. Let $S, B, C \in \mathcal{B}(\mathcal{E})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Hence*

$$\Lambda_{e,\varepsilon}(S,B,C) = \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|} \sigma_e(S + CDB).$$

Proof. If $\lambda \notin \Lambda_{e,\varepsilon}(S,B,C)$, hence for each $D \in \mathcal{B}(\mathcal{E})$ with

$$\|D\| < \varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|,$$

we have $S + CDB - \lambda I \in \Phi(\mathcal{E})$ and $ind(S + CDB - \lambda I) = 0$. Then for all $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|$, we get $\lambda \notin \sigma_e(S + CDB)$. Thus

$$\lambda \notin \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|} \sigma_e(S + CDB).$$

Consequently,

$$\bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|}\sigma_e(S+CDB)\subseteq\Lambda_{e,\varepsilon}(S,B,C).$$

Conversely, if

$$\lambda\notin\bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|}\sigma_e(S+CDB),$$

hence for each $D\in\mathcal{B}(\mathcal{E})$ with $\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|$, $\lambda\notin\sigma_e(S+CDB)$. Thus

$$S+CDB-\lambda I\in\Phi(\mathcal{E}) \text{ and } ind(S+CDB-\lambda I)=0,$$

for each $D\in\mathcal{B}(\mathcal{E})$ such that $\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|$, hence $\lambda\notin\Lambda_{e,\varepsilon}(S,B,C)$. □

The next theorem showed that the structured essential condition pseudospectra of bounded linear operators is invariant under perturbation of completely continuous linear operators on ultrametric Banach spaces over a spherically complete field $\mathbb{K}$.

Theorem 2.20. *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S,B,C\in\mathcal{B}(\mathcal{E})$ with $0\in\rho(B)\cap\rho(C)$ and $\varepsilon>0$. Hence*

$$\Lambda_{e,\varepsilon}(S,B,C)=\Lambda_{e,\varepsilon}(S+K,B,C) \text{ for each } K\in\mathcal{C}_c(\mathcal{E}). \tag{2.4}$$

Proof. If $\lambda\notin\Lambda_{e,\varepsilon}(S,B,C)$, then for each $D\in\mathcal{B}(\mathcal{E})$ with

$$\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|,$$

$S+CDB-\lambda I\in\Phi(\mathcal{E})$ and $ind(S+CDB-\lambda I)=0$. From Lemma 1.10 and Lemma 1.11, for all $K\in\mathcal{C}_c(\mathcal{E})$ and $D\in\mathcal{B}(\mathcal{E})$ with

$$\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|,$$

we obtain that

$$S+CDB+K-\lambda I\in\Phi(\mathcal{E}) \text{ and } ind(S+CDB+K-\lambda I)=0. \tag{2.5}$$

By (2.5), we get $\lambda\notin\Lambda_{e,\varepsilon}(S+K,B,C)$. Then

$$\Lambda_{e,\varepsilon}(S+K,B,C)\subseteq\Lambda_{e,\varepsilon}(S,B,C).$$

The opposite inclusion follows from symmetry.

□

Theorem 2.21. *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. Let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Hence*

$$\Lambda_{e,\varepsilon}(S, B, C) = \bigcap_{K \in \mathcal{C}_c(\mathcal{E})} \Lambda_\varepsilon(S + K, B, C).$$

Proof. If $\lambda \notin \bigcap_{K \in \mathcal{C}_c(\mathcal{E})} \Lambda_\varepsilon(S + K, B, C)$, hence there is $K \in \mathcal{C}_c(\mathcal{E})$ with

$$\lambda \notin \Lambda_\varepsilon(S + K, B, C).$$

By Theorem 2.13, for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$, we get $(S + K + CDB - \lambda I)^{-1} \in \mathcal{B}(\mathcal{E})$. Then for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$, we have

$$S + K + CDB - \lambda I \in \Phi(\mathcal{E}) \text{ and } ind(S + K + CDB - \lambda I) = 0. \tag{2.6}$$

By Lemma 1.10 and Lemma 1.11, for each $D \in \mathcal{B}(\mathcal{E})$ such that $\|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$, we get

$$S + CDB - \lambda I \in \Phi(\mathcal{E}) \text{ and } \ ind(S + CDB - \lambda I) = 0. \tag{2.7}$$

Hence $\lambda \notin \Lambda_{e,\varepsilon}(S, B, C)$. Thus

$$\Lambda_{e,\varepsilon}(S, B, C) \subseteq \bigcap_{K \in \mathcal{C}(\mathcal{E})} \Lambda_\varepsilon(S + K, B, C). \tag{2.8}$$

Conversely, if $\lambda \notin \Lambda_{e,\varepsilon}(S, B, C)$. From Theorem 2.19, for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$, $\lambda \notin \sigma_e(S + CDB)$. By Theorem 1.12, there is $K \in \mathcal{C}_c(\mathcal{E})$ with $\lambda \notin \sigma(S + CDB + K)$, hence for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|$, $\lambda \in \rho(S + K + CDB)$. Then

$$\lambda \in \bigcap_{D \in \mathcal{B}(\mathcal{E}) : \|D\| < \varepsilon\|C^{-1}(S - \lambda I)B^{-1}\|} \rho(S + K + CDB). \tag{2.9}$$

By Theorem 2.13, $\lambda \notin \Lambda_\varepsilon(S + K, B, C)$. Consequently,

$$\lambda \notin \bigcap_{K \in \mathcal{C}_c(\mathcal{E})} \Lambda_\varepsilon(S + K, B, C).$$

Thus $\Lambda_{e,\varepsilon}(S, B, C) = \bigcap_{K \in \mathcal{C}_c(\mathcal{E})} \Lambda_\varepsilon(S + K, B, C)$.

∎

Remark 2.22. Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ with $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. Let $S, B, C \in \mathcal{B}(\mathcal{E})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. From Example 3.31. of [9] and Theorem 2.21, we obtain that

$$\Lambda_{e,\varepsilon}(S, B, C) = \bigcap_{F \in \mathcal{F}_0(\mathcal{E})} \Lambda_\varepsilon(S + F, B, C),$$

where $\mathcal{F}_0(\mathcal{E})$ is the set of all finite rank operators on $\mathcal{E}$.

Proposition 2.23. *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ with $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. Let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Hence*

(i) $\Lambda_{e,\varepsilon}(S, B, C) \subset \Lambda_\varepsilon(S, B, C)$.

(ii) *For each ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$, we get*

$$\sigma_e(S) \subset \Lambda_{e,\varepsilon_1}(S, B, C) \subset \Lambda_{e,\varepsilon_2}(S, B, C).$$

Proof. (i) If $\lambda \in \Lambda_{e,\varepsilon}(S, B, C)$. From Theorem 2.19,

$$\lambda \in \bigcup_{D \in \mathcal{B}(\mathcal{E}): \|D\| < \varepsilon \|C^{-1}(S - \lambda I)B^{-1}\|} \sigma_e(S + CDB).$$

By $\sigma_e(S + CDB) \subset \sigma(S + CDB)$, hence

$$\lambda \in \bigcup_{D \in \mathcal{B}(\mathcal{E}): \|D\| < \varepsilon \|C^{-1}(S - \lambda I)B^{-1}\|} \sigma(S + CDB).$$

From Theorem 2.13, $\lambda \in \Lambda_\varepsilon(S, B, C)$.

(ii) Firstly, we prove that for each $\varepsilon > 0$, $\sigma_e(S) \subset \Lambda_{e,\varepsilon}(S, B, C)$. If $\lambda \notin \Lambda_{e,\varepsilon}(S, B, C)$, then for each $D \in \mathcal{B}(\mathcal{E})$ such that

$$\|D\| < \varepsilon \|C^{-1}(S - \lambda I)B^{-1}\|,$$

we get $\lambda I - (S + CDB) \in \Phi(\mathcal{E})$ and $ind(\lambda I - (S + CDB)) = 0$. Taking limits as $\varepsilon \to 0$, $\lambda I - S \in \Phi(\mathcal{E})$ and $ind(\lambda I - S) = 0$, thus $\lambda \notin \sigma_e(S)$. Hence

$$\sigma_e(S) \subset \Lambda_{e,\varepsilon}(S, B, C).$$

If $\lambda \notin \Lambda_{e,\varepsilon_2}(S, B, C)$, hence for each $D \in \mathcal{B}(\mathcal{E})$ such that

$$\|D\| < \varepsilon_2 \|C^{-1}(S - \lambda I)B^{-1}\|,$$

we have $\lambda I - (S + CDB) \in \Phi(\mathcal{E})$ and $ind(\lambda I - (S + CDB)) = 0$. Since $\varepsilon_1 < \varepsilon_2$, for all $D \in \mathcal{B}(\mathcal{E})$ such that $\|D\| < \varepsilon_1 \|C^{-1}(S - \lambda I)B^{-1}\|$, we have $\lambda I - (S + CDB) \in \Phi(\mathcal{E})$ and $ind(\lambda I - (S + CDB)) = 0$, thus $\lambda \notin \Lambda_{e,\varepsilon_1}(S, B, C)$. Consequently, $\lambda_{e,\varepsilon_1}(S, B, C) \subset \lambda_{e,\varepsilon_2}(S, B, C)$. $\square$

Proposition 2.24. *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. Let $S, B, C \in \mathcal{B}(\mathcal{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$, hence*

$$\sigma_e(S) = \bigcap_{\varepsilon>0} \Lambda_{e,\varepsilon}(S, B, C).$$

Proof. Suppose that $\lambda \in \bigcap_{\varepsilon>0} \Lambda_{e,\varepsilon}(S, B, C)$ and $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. By Theorem 2.21,

$$\begin{aligned} \bigcap_{\varepsilon>0} \Lambda_{e,\varepsilon}(S, B, C) &= \bigcap_{\varepsilon>0} \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \Lambda_{\varepsilon}(S + K, B, C) \\ &= \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \bigcap_{\varepsilon>0} \Lambda_{\varepsilon}(S + K, B, C). \end{aligned} \tag{2.10}$$

By (i) of Proposition 2.9, $\bigcap_{\varepsilon>0} \Lambda_{\varepsilon}(S + K, B, C) = \sigma(S + K)$. By (2.10), we get

$$\bigcap_{\varepsilon>0} \Lambda_{e,\varepsilon}(S, B, C) = \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \sigma(S + K).$$

By Theorem 1.12, $\lambda \in \sigma_e(S)$. Conversely, $\lambda \in \sigma_e(S)$. From Theorem 1.12, $\lambda \in \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \sigma(S + K)$. By (i) of Proposition 2.9, we have

$$\lambda \in \bigcap_{\varepsilon>0} \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \Lambda_{\varepsilon}(S + K, B, C).$$

Since $\|\mathcal{E}\| \subseteq |\mathbb{K}|$, by Theorem 2.13,

$$\lambda \in \bigcap_{\varepsilon>0} \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|} \bigcap_{K\in\mathcal{C}_c(E)} \sigma(S + CDB + K).$$

From Theorem 1.12, $\lambda \in \bigcap_{\varepsilon>0} \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon\|C^{-1}(S-\lambda I)B^{-1}\|} \sigma_e(S + CDB)$. By Theorem 2.19, $\lambda \in \bigcap_{\varepsilon>0} \Lambda_{e,\varepsilon}(S, B, C)$. Consequently,

$$\sigma_e(S) = \bigcap_{\varepsilon>0} \Lambda_{e,\varepsilon}(S, B, C).$$

□

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STRUCTURED CONDITION PSEUDOSPECTRA AND STRUCTURED ESSENTIAL CONDITION PSEUDOSPECTRA OF BOUNDED LINEAR OPERATORS ON ULTRAMETRIC BANACH SPACES

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طیف شبه‌شرط ساختاری و طیف شبه‌شرط ساختاری اساسی اپراتورهای خطی کراندار در فضاهای باناخ اولترامتری

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در این مقاله، طیف شبه‌شرط ساختاری و طیف شبه‌شرط ساختاری اساسی اپراتورهای خطی کراندار در فضاهای باناخ اولترامتری معرفی و مطالعه می‌شود. ما به یک توصیف از طیف شبه‌شرط ساختاری اپراتورهای خطی پیوسته دست یافته و رابطه‌ای بین طیف شبه‌شرط ساختاری و طیف شبه‌ساختاری اپراتورهای خطی پیوسته در فضاهای باناخ اولترامتری ارائه می‌دهیم. همچنین، بسیاری از توصیفات طیف شبه‌شرط ساختاری اساسی اپراتورهای خطی کراندار و مثال‌هایی در این زمینه ارائه شده است.

کلمات کلیدی: فضاهای باناخ اولترامتری، طیف شبه، طیف شبه‌شرط، اپراتورهای خطی کراندار.