Journal of Algebraic Systems, Vol. 13(No. 3): (2025), pp 113-122.
https://doi.org/10.22044/JAS.2024.13603.1759

RESULTS ON QUOTIENT NEAR-RINGS INVOLVING ADDITIVE MAPS

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Abstract. In this paper, we study the commutativity of the quotient near-ring $\mathcal{N}/\mathcal{P}$ with left multipliers and generalized derivations satisfying certain algebraic identities on $\mathcal{P}$. Furthermore, an example is given to show that the 3-primeness condition used in our results is necessary.

1. Introduction

Throughout this paper, $\mathcal{N}$ will denote a left near-ring with multiplicative center $\mathcal{Z}(\mathcal{N})$. Recall that $\mathcal{N}$ is called 3-prime if $x\mathcal{N}y = \{0\}$ for $x, y \in \mathcal{N}$ implies $x = 0$ or $y = 0$ and $\mathcal{N}$ is said to be a 2-torsion free if $\mathcal{N}$ has no element of order 2. For any pair $x, y \in \mathcal{N}$, we write $[x, y] = xy - yx$ and $x \circ y = xy + yx$ to denote the Lie product and the Jordan product, respectively. A derivation on $\mathcal{N}$ is an additive endomorphism d of $\mathcal{N}$ such that $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$ or equivalently, as noted in [16, Proposition 1], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. An additive mapping $\mathcal{D} : \mathcal{N} \longrightarrow \mathcal{N}$ is called a generalized derivation if there exists a derivation d on $\mathcal{N}$ such that $\mathcal{D}(xy) = \mathcal{D}(x)y + xd(y)$ for all $x, y \in \mathcal{N}$ or equivalently, $\mathcal{D}(xy) = xd(y) + \mathcal{D}(x)y$ for all $x, y \in \mathcal{N}$. An additive mapping $H : \mathcal{N} \longrightarrow \mathcal{N}$ is said to be a left multiplier (resp. right multiplier) if $H(xy) = H(x)y$ (resp. $H(xy) = xH(y)$) for all $x, y \in \mathcal{N}$. Thereby, if H is both a left multiplier and a right multiplier, then H is said to be a multiplier of $\mathcal{N}$.

A normal subgroup $\mathcal{P}$ of $(\mathcal{N}, +)$ is called a left ideal (resp. a right ideal) if $\mathcal{N}\mathcal{P} \subseteq \mathcal{P}$ (resp. $(x + r)y - xy \in \mathcal{P}$ for all $x, y \in \mathcal{N}, r \in \mathcal{P}$), and if $\mathcal{P}$ is both a left ideal and a right ideal, then $\mathcal{P}$ is said to be an ideal of $\mathcal{N}$.

According to [11, Definition 1(iv)], an ideal $\mathcal{P}$ is a 3-prime if for $a, b \in \mathcal{N}$, $a\mathcal{N}b \subseteq \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$. We denote by $\mathcal{N}/\mathcal{P}$ the quotient near-ring and $\mathcal{Z}(\mathcal{N}/\mathcal{P})$ its multiplicative center. Clearly, if $\mathcal{P}$ is 3-prime, then $\mathcal{N}/\mathcal{P}$ is a 3-prime near-ring.

Published online: 9 April 2024
MSC(2020): Primary: 16N60; Secondary: 16W25, 16Y30.
Keywords: Prime near-rings; Derivations; Left multipliers; Commutativity.
Received: 22 September 2023, Accepted: 26 January 2024.
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In the following, we present an example of a near-ring $\mathcal{N}$ admitting a 3-prime ideal $\mathcal{P}$.

Example 1.1. Let $\mathcal{N} = \{0, a, b, c, d, e, f, g\}$ and define the two laws " + " and " . " on $\mathcal{N}$ by:

+	0	a	b	c	d	e	f	g
0	0	a	b	c	d	e	f	g
a	a	b	c	0	e	f	g	d
b	b	c	0	a	f	g	d	e
c	c	0	a	b	g	d	e	f
d	d	g	f	e	0	c	b	a
e	e	d	g	f	a	0	c	b
f	f	e	d	g	b	a	0	c
g	g	f	e	d	c	b	a	0

and

.	0	a	b	c	d	e	f	g
0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	a	a	0
b	0	b	0	b	0	b	b	0
c	0	c	0	c	0	c	c	0
d	d	d	d	d	d	d	d	d
e	d	e	d	e	d	e	e	d
f	d	f	d	f	d	f	f	d
g	d	g	d	g	d	g	g	d

We can see that $\mathcal{N}$ is a left near-ring and $\mathcal{P} = \{0, a, b, c\}$ is a 3-prime ideal of $\mathcal{N}$.

In [14], the authors defined a special derivation $\tilde{d}$ on $\mathcal{N}/\mathcal{P}$ by $\tilde{d}(\bar{x}) = \overline{d(x)}$ for all $x \in \mathcal{N}$. Motivated by this concept, we define a left multiplier $\tilde{H}$ and a generalized derivation $\tilde{\mathcal{D}}$ on $\mathcal{N}/\mathcal{P}$ as follows: $\tilde{H}(\bar{x}) = \overline{H(x)}$ and $\tilde{\mathcal{D}}(\bar{x}) = \overline{\mathcal{D}(x)}$ for all $x \in \mathcal{N}$, respectively.

There are several results claiming that 3-prime near-rings, under certain constraints due to special maps, have ring-like behavior, see for example [4, 8, 6, 7, 13, 12], where further references can be found.

Recently, Ashraf et al. [1] proved that if a 3-prime near-ring $\mathcal{N}$ admits a nonzero derivation d satisfying $d([x, y]) = [d(x), y]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ must be a commutative ring. Motivated by this result, Enguady et al. [10] studied the commutativity of near-rings admitting a left derivation d and a multiplier H satisfying $d([x, u]) = H([x, u])$ for all $x \in \mathcal{N}$, $u \in U$, where U is a Lie ideal of $\mathcal{N}$.

In this note, we investigate the commutativity of a near-ring $\mathcal{N}/\mathcal{P}$ in the case when $\mathcal{N}$ admitting a generalized derivation $\mathcal{D}$ and a left multiplier H satisfying any one of the properties (i) $\mathcal{D}([x, y]) - H([x, y]) \in \mathcal{P}$, (ii) $\mathcal{D}(x \circ y) - H(x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$, where $\mathcal{P}$ is a 3-prime ideal of $\mathcal{N}$. The obtained results improve and unify those proved in [5, Theorem 2.2] and [1, Theorem 1(i)].

2. Main results

To facilitate our discussion, we need the following lemmas.

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring.*

a) [2, Lemma 1.3(i)] *If x is an element of $\mathcal{N}$ such that $\mathcal{N}x = \{0\}$ (resp. $x\mathcal{N} = \{0\}$), then $x = 0$.*
b) [2, Lemma 1.5)] *If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*
c) [9, Lemma 2] *Let d be a derivation on $\mathcal{N}$. If $x \in Z(\mathcal{N})$, then $d(x) \in Z(\mathcal{N})$.*
d) [2, Theorem 2.1] *If $\mathcal{N}$ admits a nonzero derivation d for which $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

The next lemma gives partial distributivity to the right of the multiplicative law. In our proof, we frequently use this result without mentioning it.

Lemma 2.2. [3, Lemma 1.3] *Let $\mathcal{N}$ be a near-ring. If $\mathcal{N}$ admits a generalized derivation $\mathcal{D}$ associated with a derivation d, then*

$$(\mathcal{D}(x)y + xd(y))z = \mathcal{D}(x)yz + xd(y)z \text{ for all } x, y, z \in \mathcal{N}.$$

Theorem 2.3. *Let $\mathcal{P}$ be a 3-prime ideal of a near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a generalized derivation $\mathcal{D}$ associated with a derivation d for which $d(\mathcal{N}) \nsubseteq \mathcal{P}$, and a left multiplier H, then the following assertions are equivalent:*

i) $\mathcal{D}([x, y]) - H([x, y]) \in \mathcal{P}$ *for all $x, y \in \mathcal{N}$,*
ii) $\mathcal{N}/\mathcal{P}$ *is a commutative ring.*

Proof. Obviously, we have $(ii) \Rightarrow (i)$. So, we need to prove that $(i) \Rightarrow (ii)$. By hypotheses given, we have $\mathcal{D}([x, y]) - H([x, y]) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$, which implies that

$$\tilde{\mathcal{D}}([\bar{x}, \bar{y}]) = \tilde{H}([\bar{x}, \bar{y}]) \text{ for all } x, y \in \mathcal{N}. \tag{2.1}$$

Case 1: Suppose that $H(\mathcal{N}) \subseteq \mathcal{P}$, so that (2.1) yields

$$\tilde{\mathcal{D}}([\bar{x}, \bar{y}]) = \bar{0} \text{ for all } x, y \in \mathcal{N}. \tag{2.2}$$

Substituting $[u, v]$ and $[u, v]y$ for x and y, respectively, in (2.2) and using it again, we obtain

$$\begin{aligned} \tilde{\mathcal{D}}([[\bar{u}, \bar{v}], [\bar{u}, \bar{v}]\bar{y}]) &= \tilde{\mathcal{D}}([\bar{u}, \bar{v}][[\bar{u}, \bar{v}], \bar{y}]) \\ &= \tilde{\mathcal{D}}([\bar{u}, \bar{v}])[[\bar{u}, \bar{v}], \bar{y}] + [\bar{u}, \bar{v}]\tilde{d}([[\bar{u}, \bar{v}], \bar{y}]) \\ &= \bar{0} + [\bar{u}, \bar{v}]\tilde{d}([[\bar{u}, \bar{v}], \bar{y}]). \end{aligned}$$

It follows that

$$[\bar{u}, \bar{v}]\tilde{d}([[\bar{u}, \bar{v}], \bar{y}]) = \bar{0} \text{ for all } u, v, y \in \mathcal{N}. \tag{2.3}$$

Also, putting $[u,v]y$ instead of y in (2.3) and using (2.3) we find that $[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])[[\bar{u},\bar{v}],\bar{y}]=\bar{0}$ for all $u,v,y\in\mathcal{N}$ which means that

$$[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])[\bar{u},\bar{v}]\bar{y}=[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])\bar{y}[\bar{u},\bar{v}] \text{ for all } u,v,y\in\mathcal{N}. \tag{2.4}$$

Taking yt instead of y in (2.4) and using it, we obtain $[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])\bar{y}[[\bar{u},\bar{v}],\bar{t}\,]=\bar{0}$ for all $u,v,y,t\in\mathcal{N}$ which reduces to

$$[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])\,\mathcal{N}/\mathcal{P}\,[[\bar{u},\bar{v}],\bar{t}\,]=\{\bar{0}\} \text{ for all } u,v,t\in\mathcal{N}. \tag{2.5}$$

In view of the 3-primeness of $\mathcal{N}/\mathcal{P}$, (2.5) assures that

$$[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])=\bar{0} \text{ or } [\bar{u},\bar{v}]\in Z(\mathcal{N}/\mathcal{P}) \text{ for all } u,v\in\mathcal{N}. \tag{2.6}$$

Our aim is to show that (2.6) gives $[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])=\bar{0}$ for all $u,v\in\mathcal{N}$. Indeed, suppose that there exist $u_0,v_0\in\mathcal{N}$ such that $[\bar{u_0},\bar{v_0}]\in Z(\mathcal{N}/\mathcal{P})$. Return to (2.2) and replacing x and y by $[u_0,v_0]u_0$ and v_0 respectively, we find that $[\bar{u_0},\bar{v_0}]\tilde{d}([\bar{u_0},\bar{v_0}])=\bar{0}$. Consequently, (2.6) reduces to

$$[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])=\bar{0} \text{ for all } u,v\in\mathcal{N}. \tag{2.7}$$

From (2.3), we have $[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}]\bar{y})=[\bar{u},\bar{v}]\tilde{d}(\bar{y}[\bar{u},\bar{v}])$ for $u,v,y\in\mathcal{N}$. Solving this equation and using (2.7), we obtain

$$[\bar{u},\bar{v}]^2\tilde{d}(\bar{y})=[\bar{u},\bar{v}]\tilde{d}(\bar{y})[\bar{u},\bar{v}]+[\bar{u},\bar{v}]\bar{y}\tilde{d}([\bar{u},\bar{v}]) \text{ for all } u,v,y\in\mathcal{N}. \tag{2.8}$$

Taking $y=y[u,v]$ in (2.8) and using (2.7), we find that

$$\begin{aligned}[\bar{u},\bar{v}]^2\tilde{d}(\bar{y}[\bar{u},\bar{v}]) &= [\bar{u},\bar{v}]\tilde{d}(\bar{y}[\bar{u},\bar{v}])[\bar{u},\bar{v}]+[\bar{u},\bar{v}]\bar{y}[\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])\\ &= [\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}]\bar{y})[\bar{u},\bar{v}] \text{ for all } u,v,y\in\mathcal{N}.\end{aligned}$$

By the definition of $\tilde{d}$, and after simplification, we obtain $[\bar{u},\bar{v}]^2\bar{y}\tilde{d}([\bar{u},\bar{v}])=\bar{0}$ for all $u,v,y\in\mathcal{N}$ and therefore,

$$[\bar{u},\bar{v}]^2\,\mathcal{N}/\mathcal{P}\,\tilde{d}([\bar{u},\bar{v}])=\{\bar{0}\} \text{ for all } u,v\in\mathcal{N}. \tag{2.9}$$

Since $\mathcal{N}/\mathcal{P}$ is 3-prime, (2.9) shows that

$$[\bar{u},\bar{v}]^2=\bar{0} \;\text{ or }\; \tilde{d}([\bar{u},\bar{v}])=\bar{0} \text{ for all } u,v\in\mathcal{N}. \tag{2.10}$$

Let $u_0,v_0\in\mathcal{N}$ such that $[\bar{u_0},\bar{v_0}]^2=\bar{0}$. On the one hand, (2.8) gives

$$[\bar{u_0},\bar{v_0}]\tilde{d}(\bar{y})[\bar{u_0},\bar{v_0}]+[\bar{u_0},\bar{v_0}]\bar{y}\tilde{d}([\bar{u_0},\bar{v_0}])=\bar{0} \text{ for all } y\in\mathcal{N}. \tag{2.11}$$

On the other hand, by virtue of (2.2) we have also $\tilde{\mathcal{D}}([[\bar{u_0},\bar{v_0}],\bar{y}])=\bar{0}$ for all $y\in\mathcal{N}$ which implies that $\tilde{\mathcal{D}}([\bar{u_0},\bar{v_0}]\bar{y})=\tilde{\mathcal{D}}(\bar{y}[\bar{u_0},\bar{v_0}])$ for all $y\in\mathcal{N}$. This implies that

$$[\bar{u_0},\bar{v_0}]\tilde{d}(\bar{y})=\tilde{\mathcal{D}}(\bar{y})[\bar{u_0},\bar{v_0}]+\bar{y}\tilde{d}([\bar{u_0},\bar{v_0}]) \text{ for all } y\in\mathcal{N}.$$

Right-multiplying this equation by $[\bar{u_0}, \bar{v_0}]$ and using Lemma 2.2, we obtain for all $y \in \mathcal{N}$

$$[\bar{u_0}, \bar{v_0}]\tilde{d}(\bar{y})[\bar{u_0}, \bar{v_0}] = \tilde{\mathcal{D}}(\bar{y})[\bar{u_0}, \bar{v_0}]^2 + \bar{y}\tilde{d}([\bar{u_0}, \bar{v_0}])[\bar{u_0}, \bar{v_0}]. \tag{2.12}$$

According to (2.7) and the additivity of $\tilde{d}$, we obtain

$$\tilde{d}([\bar{u_0}, \bar{v_0}]^2) = \bar{0} = \tilde{d}([\bar{u_0}, \bar{v_0}])[\bar{u_0}, \bar{v_0}],$$

and therefore (2.12) reduces to $[\bar{u_0}, \bar{v_0}]\tilde{d}(\bar{y})[\bar{u_0}, \bar{v_0}] = \bar{0}$ for all $y \in \mathcal{N}$. Consequently, (2.11) implies that

$$[\bar{u_0}, \bar{v_0}]\,\mathcal{N}/\mathcal{P}\,\tilde{d}([\bar{u_0}, \bar{v_0}]) = \{\bar{0}\}$$

which, in view of the 3-primeness of $\mathcal{N}/\mathcal{P}$, forces that $[\bar{u_0}, \bar{v_0}] = \bar{0}$ or $\tilde{d}([\bar{u_0}, \bar{v_0}]) = \bar{0}$; but the first condition yields also $\tilde{d}([\bar{u_0}, \bar{v_0}]) = \bar{0}$ and hence (2.7) shows that $\tilde{d}([\bar{u}, \bar{v}]) = \bar{0}$ for all $u, v \in \mathcal{N}$ which, taking account the special case when $\alpha = id_{\mathcal{N}}$ in [15, Theorem 2.3], implies that $\mathcal{N}/\mathcal{P}$ is a commutative ring.

Case 2: Suppose that $H(\mathcal{N}) \nsubseteq \mathcal{P}$. In this case, replacing x and y by $[u, v]x$ and $[u, v]$, respectively, in (2.1) and using it, we obtain

$$[\bar{u}, \bar{v}]\tilde{d}([\bar{x}, [\bar{u}, \bar{v}]]) = \bar{0} \text{ for all } u, v, x \in \mathcal{N}. \tag{2.13}$$

Next, taking $x = [u, v]x$ in (2.13) we find that

$$\begin{aligned} \bar{0} &= [\bar{u}, \bar{v}]\tilde{d}\Big([\bar{u}, \bar{v}][\bar{x}, [\bar{u}, \bar{v}]]\Big) \\ &= [\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}])[\bar{x}, [\bar{u}, \bar{v}]] + [\bar{u}, \bar{v}][\bar{u}, \bar{v}]\tilde{d}([\bar{x}, [\bar{u}, \bar{v}]]) \\ &= [\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}])[\bar{x}, [\bar{u}, \bar{v}]] \end{aligned} \tag{2.14}$$

From (2.14), it follows that $[\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}])\bar{x}[\bar{u}, \bar{v}] = [\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}])[\bar{u}, \bar{v}]\bar{x}$ for all $u, v, x \in \mathcal{N}$. Substituting xt for x in the last equation and using it again, we infer that $[\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}])\bar{x}[\bar{t}, [\bar{u}, \bar{v}]] = \bar{0}$ for all $x, t, u, v \in \mathcal{N}$. Accordingly, $[\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}])\,\mathcal{N}/\mathcal{P}\,[\bar{t}, [\bar{u}, \bar{v}]] = \{\bar{0}\}$ for all $t, u, v \in \mathcal{N}$. Using the 3-primeness of $\mathcal{N}/\mathcal{P}$, we conclude that

$$[\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}]) = \bar{0} \text{ or } [\bar{u}, \bar{v}] \in Z(\mathcal{N}/\mathcal{P}) \text{ for all } u, v \in \mathcal{N}. \tag{2.15}$$

Let $u_0, v_0 \in \mathcal{N}$ such that $[\bar{u_0}, \bar{v_0}] \in Z(\mathcal{N}/\mathcal{P})$. In (2.1), replacing x and y by $[u_0, v_0]u_0$ and v_0 respectively, we arrive at $[\bar{u_0}, \bar{v_0}]\tilde{d}([\bar{u_0}, \bar{v_0}]) = \bar{0}$, so that (2.15) reduces to

$$[\bar{u}, \bar{v}]\tilde{d}([\bar{u}, \bar{v}]) = \bar{0} \text{ for all } u, v \in \mathcal{N}. \tag{2.16}$$

Also, from (2.13), we have $[\bar{u},\bar{v}]\tilde{d}(\bar{x}[\bar{u},\bar{v}]) = [\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}]\bar{x})$ for all $x,u,v \in \mathcal{N}$. Solving this equation and invoking (2.16), we obtain

$$[\bar{u},\bar{v}]\bar{x}\tilde{d}([\bar{u},\bar{v}]) + [\bar{u},\bar{v}]\tilde{d}(\bar{x})[\bar{u},\bar{v}] = [\bar{u},\bar{v}]^2\tilde{d}(\bar{x}) \text{ for all } x,u,v \in \mathcal{N}. \tag{2.17}$$

In (2.17), putting $x[u,v]$ instead of x and invoking (2.16), we find that for all $x,u,v \in \mathcal{N}$.

$$[\bar{u},\bar{v}]\bar{x}\tilde{d}([\bar{u},\bar{v}])[\bar{u},\bar{v}] + [\bar{u},\bar{v}]\tilde{d}(\bar{x})[\bar{u},\bar{v}]^2 = [\bar{u},\bar{v}]^2\tilde{d}(\bar{x})[\bar{u},\bar{v}] + [\bar{u},\bar{v}]^2\bar{x}\tilde{d}([\bar{u},\bar{v}]) \tag{2.18}$$

Combining (2.17) and (2.18) and after simplifying, we get $[\bar{u},\bar{v}]^2\bar{x}\tilde{d}([\bar{u},\bar{v}]) = \bar{0}$ for all $x,u,v \in \mathcal{N}$ which, in view of the 3-primeness of $\mathcal{N}/\mathcal{P}$, implies that

$$[\bar{u},\bar{v}]^2 = \bar{0} \text{ or } \tilde{d}([\bar{u},\bar{v}]) = \bar{0} \text{ for all } x,u,v \in \mathcal{N}. \tag{2.19}$$

Suppose that there exist $u_0, v_0 \in \mathcal{N}$ such that $[\bar{u_0},\bar{v_0}]^2 = \bar{0}$. In this case, returning to (2.1) and replacing x by $[u_0,v_0]$, we get

$$\tilde{\mathcal{D}}([[\bar{u_0},\bar{v_0}],\bar{y}]) = \tilde{H}([[\bar{u_0},\bar{v_0}],\bar{y}])$$

for all $y \in \mathcal{N}$. That is, $\tilde{\mathcal{D}}([\bar{u_0},\bar{v_0}]\bar{y}) - \tilde{\mathcal{D}}(\bar{y}[\bar{u_0},\bar{v_0}]) = \tilde{H}([\bar{u_0},\bar{v_0}]\bar{y}) - \tilde{H}(\bar{y}[\bar{u_0},\bar{v_0}])$. Using the property defining of $\tilde{\mathcal{D}}$ and $\tilde{H}$ together (2.1), we find that

$$[\bar{u_0},\bar{v_0}]\tilde{d}(\bar{y}) - \bar{y}\tilde{d}([\bar{u_0},\bar{v_0}]) - \tilde{\mathcal{D}}(\bar{y})[\bar{u_0},\bar{v_0}] = -\tilde{H}(\bar{y})[\bar{u_0},\bar{v_0}] \text{ for all } y \in \mathcal{N}.$$

Replacing y by $[x,y]$ in the previous equation and left-multiplying it by $[\bar{u_0},\bar{v_0}]$, in virtue of our statement assumption, we get $[\bar{u_0},\bar{v_0}]\bar{y}\tilde{d}([\bar{u_0},\bar{v_0}]) = \bar{0}$ for all $y \in \mathcal{N}$, and hence $[\bar{u_0},\bar{v_0}]\mathcal{N}/\mathcal{P}\tilde{d}([\bar{u_0},\bar{v_0}]) = \{\bar{0}\}$. As $\mathcal{N}/\mathcal{P}$ is 3-prime and $\tilde{d}$ is additive, the last expression assures that $\tilde{d}([\bar{u_0},\bar{v_0}]) = \bar{0}$ and thus (2.19) leads to $\tilde{d}([\bar{u},\bar{v}]) = \bar{0}$ for all $u,v \in \mathcal{N}$, which forces that $\mathcal{N}/\mathcal{P}$ is a commutative ring by [15, Theorem 2.3].

□

The next theorem generalizes the result [5, Theorem 2.4].

Theorem 2.4. *Let $\mathcal{N}$ be a 2-torsion near-ring, $\mathcal{P}$ be a 3-prime ideal of $\mathcal{N}$, and H be a left multiplier of $\mathcal{N}$. Then, $\mathcal{N}$ admits no generalized derivation $\mathcal{D}$ associated with a derivation d for which $d(\mathcal{N}) \nsubseteq \mathcal{P}$ and $\mathcal{D}(x \circ y) - H(x \circ y) \in \mathcal{P}$ for all $x,y \in \mathcal{N}$.*

Proof. By hypotheses given, we have

$$\tilde{\mathcal{D}}(\bar{x} \circ \bar{y}) = \tilde{H}(\bar{x} \circ \bar{y}) \text{ for all } x,y \in \mathcal{N}. \tag{2.20}$$

Replacing y by xy in (2.12), we obtain $\tilde{\mathcal{D}}(\bar{x}(\bar{x} \circ \bar{y})) = \tilde{H}(\bar{x}(\bar{x} \circ \bar{y}))$ for all $x,y \in \mathcal{N}$. Again, substituting $u \circ v$ for x in the last equation and applying

(2.12), we arrive at $(\bar{u}\circ\bar{v})\tilde{d}((\bar{u}\circ\bar{v})\circ\bar{y})=\bar{0}$ for all $u,v,y\in\mathcal{N}$. Again, taking $(u\circ v)y$ instead of y we infer that $(\bar{u}\circ\bar{v})\tilde{d}(\bar{u}\circ\bar{v})((\bar{u}\circ\bar{v})\circ\bar{y})=\bar{0}$. So that, for all $u,v,y\in\mathcal{N}$, we have

$$(\bar{u}\circ\bar{v})\tilde{d}(\bar{u}\circ\bar{v})((\bar{u}\circ\bar{v})\bar{y})=-(\bar{u}\circ\bar{v})\tilde{d}(\bar{u}\circ\bar{v})\bar{y}(\bar{u}\circ\bar{v}). \tag{2.21}$$

Substituting yt for y in (2.21) and using (2.21), we get

$$(\bar{u}\circ\bar{v})\tilde{d}(\bar{u}\circ\bar{v})\bar{y}[(\bar{u}\circ\bar{v}),\,\tilde{t}\,]=\bar{0}$$

for all $u,v,y,t\in\mathcal{N}$ which can be written as

$$(\bar{u}\circ\bar{v})\tilde{d}(\bar{u}\circ\bar{v})\,\mathcal{N}/\mathcal{P}\,[-(\bar{u}\circ\bar{v}),\,\tilde{t}\,]=\{\bar{0}\}.$$

In virtue of the 3-primeness of $\mathcal{N}/\mathcal{P}$, the preceding result shows that

$$(\bar{u}\circ\bar{v})\tilde{d}(\bar{u}\circ\bar{v})=\bar{0} \text{ or } -(\bar{u}\circ\bar{v})\in Z(\mathcal{N}/\mathcal{P}) \text{ for all } u,v,y\in\mathcal{N}. \tag{2.22}$$

Letting u_0,v_0 two elements of $\mathcal{N}$ such that $-(\bar{u_0}\circ\bar{v_0})\in Z(\mathcal{N}/\mathcal{P})$. In particular, by replacing x by $(-(u_0\circ v_0))x$ in (2.20) and using (2.20), we find that $(-(u_0\circ v_0))\tilde{d}(\bar{x}\circ\bar{y})=\bar{0}$ for all $x,y\in\mathcal{N}$. Left-multiplying this equation by $\bar{r}$, where $r\in\mathcal{N}$, and using the fact that $\mathcal{N}/\mathcal{P}$ is 3-prime, we conclude that $\bar{u_0}\circ\bar{v_0}=\bar{0}$ or $\tilde{d}(\bar{x}\circ\bar{y})=\bar{0}$ for all $x,y\in\mathcal{N}$. But, in view of [15, Corollary 3.7], the second condition cannot be verified and therefore, (2.22) gives

$$(\bar{u}\circ\bar{v})\tilde{d}(\bar{u}\circ\bar{v})=\bar{0} \text{ for all } u,v\in\mathcal{N}. \tag{2.23}$$

Now, return to (2.20) and taking $x=u\circ v$ and $y=(u\circ v)y$, we find that $(\bar{u}\circ\bar{v})\tilde{d}((\bar{u}\circ\bar{v})\circ\bar{y})=\bar{0}$ which implies that

$$(\bar{u}\circ\bar{v})\tilde{d}((\bar{u}\circ\bar{v})\bar{y})=-(\bar{u}\circ\bar{v})\tilde{d}(\bar{y}(\bar{u}\circ\bar{v})) \text{ for all } u,v,y\in\mathcal{N}.$$

Solving this equation and using (2.23), we obtain for all $u,v,y\in\mathcal{N}$,

$$(\bar{u}\circ\bar{v})^2\tilde{d}(\bar{y}) = -(\bar{u}\circ\bar{v})\bar{y}\tilde{d}(\bar{u}\circ\bar{v})-(\bar{u}\circ\bar{v})\tilde{d}(\bar{y})(\bar{u}\circ\bar{v}) \tag{2.24}$$

$$= -(\bar{u}\circ\bar{v})\tilde{d}(\bar{y})(\bar{u}\circ\bar{v})-(\bar{u}\circ\bar{v})\bar{y}\tilde{d}(\bar{u}\circ\bar{v}). \tag{2.25}$$

Substituting $y(u\circ v)$ for y in (2.24), we get

$$(\bar{u}\circ\bar{v})^2\tilde{d}(\bar{y})(\bar{u}\circ\bar{v})+(\bar{u}\circ\bar{v})^2\bar{y}\tilde{d}(\bar{u}\circ\bar{v}) = -(\bar{u}\circ\bar{v})\tilde{d}(\bar{y})(\bar{u}\circ\bar{v})^2 -(\bar{u}\circ\bar{v})\bar{y}\tilde{d}(\bar{u}\circ\bar{v})(\bar{u}\circ\bar{v}). \tag{2.26}$$

Right-multiplying (2.25) by $\bar{u}\circ\bar{v}$, we obtain

$$(\bar{u}\circ\bar{v})^2\tilde{d}(\bar{y})(\bar{u}\circ\bar{v})=-(\bar{u}\circ\bar{v})\tilde{d}(\bar{y})(\bar{u}\circ\bar{v})^2-(\bar{u}\circ\bar{v})\bar{y}\tilde{d}(\bar{u}\circ\bar{v})(\bar{u}\circ\bar{v}). \tag{2.27}$$

From (2.26) and (2.27), we conclude that

$$(\bar{u}\circ\bar{v})^2\bar{y}\tilde{d}(\bar{u}\circ\bar{v})=\bar{0} \text{ for all } u,v,y\in\mathcal{N}$$

that is, $(\bar{u}\circ\bar{v})^2\,\mathcal{N}/\mathcal{P}\,\tilde{d}(\bar{u}\circ\bar{v})=\{\bar{0}\}$ for all $u,v\in\mathcal{N}$. Since $\mathcal{N}/\mathcal{P}$ is 3-prime, it follows that

$$(\bar{u}\circ\bar{v})^2=\bar{0} \text{ or } \tilde{d}(\bar{u}\circ\bar{v})=\bar{0} \text{ for all } u,v\in\mathcal{N}. \tag{2.28}$$

Let $u_0,v_0\in\mathcal{N}$ such that $(\bar{u_0}\circ\bar{v_0})^2=\bar{0}$. In this case, return to (2.20) and using the additivity property of $\tilde{\mathcal{D}}$ and $\tilde{H}$, we obtain

$$\tilde{\mathcal{D}}(\bar{x}\bar{y})+\tilde{\mathcal{D}}(\bar{y}\bar{x})=\tilde{H}(\bar{x}\bar{y})+\tilde{H}(\bar{y}\bar{x})$$

for all $x,y\in\mathcal{N}$. And thus,

$$\tilde{\mathcal{D}}(\bar{x})\bar{y}+\bar{x}\tilde{d}(\bar{y})+\bar{y}\tilde{d}(\bar{x})+\tilde{\mathcal{D}}(\bar{y})\bar{x}=\tilde{H}(\bar{x})\bar{y}+\tilde{H}(\bar{y})\bar{x} \text{ for all } x,y\in\mathcal{N}.$$

Now, by replacing x and y by $u_0\circ v_0$ and $r\circ s$ respectively in the latter equation, we get

$$(\bar{u_0}\circ\bar{v_0})\tilde{d}(\bar{r}\circ\bar{s})+(\bar{r}\circ\bar{s})\tilde{d}(\bar{u_0}\circ\bar{v_0})=\bar{0} \text{ for all } r,s\in\mathcal{N}.$$

Left-multiplying by $(\bar{u_0}\circ\bar{v_0})$ and using our statement assumption, we obtain $(\bar{u_0}\circ\bar{v_0})(\bar{r}\circ\bar{s})\tilde{d}(\bar{u_0}\circ\bar{v_0})=\bar{0}$ for all $r,s\in\mathcal{N}$. It follows that $\Big((\bar{u_0}\circ\bar{v_0})\bar{r}\bar{s}+(\bar{u_0}\circ\bar{v_0})\bar{s}\bar{r}\Big)\tilde{d}(\bar{u_0}\circ\bar{v_0})=\bar{0}$ for all $r,s\in\mathcal{N}$. Again, taking $s=(u_0\circ v_0)s$, we find that $(\bar{u_0}\circ\bar{v_0})\,\bar{r}\,(\bar{u_0}\circ\bar{v_0})\,\bar{s}\,\tilde{d}(\bar{u_0}\circ\bar{v_0})=\bar{0}$ for all $r,s\in\mathcal{N}$, which can be written as

$$(\bar{u_0}\circ\bar{v_0})\,\mathcal{N}/\mathcal{P}\,(\bar{u_0}\circ\bar{v_0})\,\mathcal{N}/\mathcal{P}\,\tilde{d}(\bar{u_0}\circ\bar{v_0})=\{\bar{0}\}. \tag{2.29}$$

In the light of the 3-primeness of $\mathcal{N}/\mathcal{P}$, (2.29) implies that $\tilde{d}(\bar{u_0}\circ\bar{v_0})=\bar{0}$ and hence (2.28) reduces to $\tilde{d}(\bar{u}\circ\bar{v})=\bar{0}$ for all $u,v\in\mathcal{N}$. In view of [15, Corollary 3.7], we get the required result.

□

The following example proves that the 3-primeness of $\mathcal{P}$ that we used in our theorems is necessary.

Example 2.5. Consider $\mathcal{M}$ be an any left near-ring and let us define $\mathcal{N},\mathcal{P},\mathcal{D}$, d,H by:

$$\mathcal{N}=\left\{\begin{pmatrix}0 & r & 0\\ 0 & 0 & 0\\ 0 & s & t\end{pmatrix} \mid 0,r,s,t\in\mathcal{M}\right\},\ \mathcal{P}=\left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & r & 0\end{pmatrix} \mid 0,r\in\mathcal{M}\right\},$$

$$d\begin{pmatrix}0 & r & 0\\ 0 & 0 & 0\\ 0 & s & t\end{pmatrix}=\begin{pmatrix}0 & r & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ H\begin{pmatrix}0 & r & 0\\ 0 & 0 & 0\\ 0 & s & t\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & s & t\end{pmatrix}$$

and $\mathcal{D} = H$. We can see that $\mathcal{P}$ is a non-3-prime ideal of the near-ring $\mathcal{N}$, $\mathcal{D}$ is a generalized derivation of $\mathcal{N}$ associated with a derivation d, and H is a multiplier of $\mathcal{N}$ which satisfies all hypotheses of our theorems. Furthermore, $\mathcal{N}/\mathcal{P}$ is not a commutative ring.

Acknowledgments

The authors thank the reviewer for valuable suggestions and comments.

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Journal of Algebraic Systems

RESULTS ON QUOTIENT NEAR-RINGS INVOLVING ADDITIVE MAPS

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نتایجی درباره شبه‌حلقه‌های خارج‌قسمتی در ارتباط با نگاشت‌های جمعی

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در این مقاله، جابجایی‌پذیری شبه‌حلقه خارج‌قسمتی $\mathcal{N}/\mathcal{P}$ با مضرب‌های چپ و مشتقات تعمیم‌یافته که در مشخصه‌های جبری خاصی روی $\mathcal{P}$ صدق می‌کنند، را بررسی می‌کنیم. به علاوه، مثالی ارائه شده که نشان می‌دهد شرط ۳-اولیه که در نتایج ما استفاده شده است، ضروری می‌باشد.

کلمات کلیدی: شبه‌حلقه‌های اول، مشتقات، مضرب‌های چپ، جابجایی‌پذیری.