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JNB-ALGEBRAS

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Abstract. As a generalization of the self-distributive BE-algebra, the JNB-algebra is introduced, and its basic properties are investigated. This could play various roles in the study of logical algebra, including BCK-algebra. First, examples are presented showing that the three axioms of JNB-algebra are independent of each other. The basic properties of JNB-algebras that will be needed to study various theories about JNB-algebras are explored. Upper sets based on one and two elements are introduced and its associated properties are examined. Two concepts so called JNB-deductive system and JNB-filter are introduced, and their properties are investigated. Characterizations of the JNB-deductive system and the JNB-filter are discussed. It is finally confirmed that the JNB-deductive system matches the JNB-filter.

1. Introduction

L. Henkin and T. Skolem made significant contributions to the field of intuitionistic and non-classical logics during the 1950s by introducing Hilbert algebras. A. Diego studied the local finiteness of Hilbert algebras (see [9]). Later, several researchers introduced different concepts in Hilbert algebras and studied its properties (see [8, 7, 10]). J. C. Abbott introduced a concept of implication algebra in the sake to formalize the logical connective implication in the classical propositional logic and has shown that implication algebras are a natural generalization of Boolean algebras (see [1]). R. A. Borzooei and S. K. Shoar have shown that the implication algebras are equivalent to dual implicative BCK-algebras (see [5]). H. S. Kim and Y. H. Kim introduced the notion of BE-algebras as a generalization of dual BCK-algebras (see [11]). A. Rezaei et al. studied the relationship between Hilbert algebras and BE-algebras (see [12]). S. S. Ahn et al. studied the notions of ideas and upper sets in BE-algebras (see [2]). A. Borumand Saeid et al. introduced and studied some types of filters in BE-algebras (see [4]). R. A. Borzooei and J. Shohani introduced the notion of a generalized Hilbert algebra and studied its properties (see [6]). The process of generalization is pivotal in the study

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of algebraic structures, leading to the introduction of GE-algebras by R. K. Bandaru et al. elaborated in (see [3]).

In this paper, we introduce the concept of a JNB-algebra which is a generalization of self-distributive BE-algebra and study its properties. We give the relation between other algebraic structures related to BCK-algebra. We introduce the notion of upper sets in a JNB-algebra and investigate its properties. We consider JNB-deductive systems and JNB-filters in a JNB-algebra and characterize both.

2. Preliminaries

Diego (cf. [9]) introduced the Hilbert algebra as follows.

Definition 2.1 ([9]). A *Hilbert algebra* is defined to be an algebra $(X, *, 1)$ of type (2,0) satisfying the following conditions:
(H1) $\mathfrak{a} * (\mathfrak{b} * \mathfrak{a}) = 1$,
(H2) $(\mathfrak{a} * (\mathfrak{b} * \mathfrak{c})) * ((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) = 1$,
(H3) $\mathfrak{a} * \mathfrak{b} = 1 = \mathfrak{b} * \mathfrak{a} \Rightarrow \mathfrak{a} = \mathfrak{b}$
for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Definition 2.2 ([11]). A *BE-algebra* is an algebra $(X, *, 1)$ of type $(2, 0)$ that satisfies:
(BE1) $\mathfrak{a} * \mathfrak{a} = 1$,
(BE2) $1 * \mathfrak{a} = \mathfrak{a}$,
(BE3) $\mathfrak{a} * 1 = 1$,
(BE4) $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) = \mathfrak{b} * (\mathfrak{a} * \mathfrak{c})$,
for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

A BE-algebra $(X, *, 1)$ is said to be *self-distributive* (see [2, Definition 2.4]) if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)(\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) = (\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})). \tag{2.1}$$

Definition 2.3 ([5]). A *dual BCK-algebra* is a triple $(X, *, 1)$ where X is a set with a constant 1 and $*$ is a binary operation on X that satisfies:
(DBCK1) $(\mathfrak{b} * \mathfrak{c}) * ((\mathfrak{c} * \mathfrak{a}) * (\mathfrak{b} * \mathfrak{a})) = 1$,
(DBCK2) $\mathfrak{b} * ((\mathfrak{b} * \mathfrak{a}) * \mathfrak{a}) = 1$,
(DBCK3) $\mathfrak{a} * \mathfrak{a} = 1$,
(DBCK4) $\mathfrak{a} * \mathfrak{b} = 1 = \mathfrak{b} * \mathfrak{a}$ imply $\mathfrak{a} = \mathfrak{b}$,
(DBCK5) $\mathfrak{a} * 1 = 1$
for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Definition 2.4 ([3]). A *GE-algebra* is a structure $(X, *, 1)$ in which X is set with a constant "1" and a binary operation "$*$" satisfying the following axioms:
(GE1) $\mathfrak{a} * \mathfrak{a} = 1$,
(GE2) $1 * \mathfrak{a} = \mathfrak{a}$,
(GE3) $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) = \mathfrak{a} * (\mathfrak{b} * (\mathfrak{a} * \mathfrak{c}))$
for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

3. JNB-algebras

In this section, we give the definition of a JNB-algebra and its properties.

Definition 3.1. A *JNB-algebra* is defined to be an algebra $(X, *, 1)$, where X is a set with a special element 1 and $*$ is a binary operation on X that satisfies:

$$1 * x = x, \tag{3.1}$$
$$x * (y * z) = (y * x) * (y * z), \tag{3.2}$$
$$(x * y) * ((y * z) * (x * z)) = 1 \tag{3.3}$$

for all $x, y, z \in X$.

We first illustrate how the three axioms of JNB algebra are independent of each other through examples.

Example 3.2. (i) Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	c
c	1	1	c	c

At this time, $(X, *, 1)$ satisfies (3.1) and (3.2), but not satisfy (3.3) because of $(a * a) * ((a * b) * (a * b)) = 1 * c = c \neq 1$.

(ii) Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	1	1	a
c	1	1	1	1

Then $(X, *, 1)$ satisfies (3.1) and (3.3), but not satisfy (3.2) since

$$b * (a * c) = b * b = 1 \neq b = 1 * b = (a * b) * (a * c).$$

(iii) Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	c	1
a	1	1	1	1
b	1	a	1	1
c	1	a	1	1

Then $(X, *, 1)$ satisfies (3.2) and (3.3), but not satisfy (3.1) because of $1 * b = c \neq b$.

We display the relationship between JNB-algebra, self-distributive BE-algebra, dual BCK-algebra, BE-algebra, and GE-algebra.

First, we show by example that the BE-algebra and the JNB-algebra are independent of each other.

Example 3.3. (i) Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	c
c	1	a	b	1

Then $(X, *, 1)$ is a BE-algebra but not a JNB-algebra because

$$a * (a * b) = a * a = 1 \neq a = 1 * a = (a * a) * (a * b).$$

(ii) Let $X = \{1, a, b, c, d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

Then $(X, *, 1)$ is a JNB-algebra. We can observe that

$$(a * b) * (a * c) = d \neq c = a * d = a * (b * c)$$

and

$$b * (a * c) = b * c = d \neq c = a * d = a * (b * c).$$

Hence $(X, *, 1)$ is not a (self-distributive) BE-algebra.

Throrugh Example 3.3(ii), we can see that the JNB-algebra is not a (self-distributive) BE-algebra.

Theorem 3.4. *Every self-distributive BE-algebra is a JNB-algebra.*

Proof. Let $(X, *, 1)$ be a self-distributive BE-algebra and $x, y, z \in X$. Then

$$x * (y * z) \overset{(BE4)}{=} y * (x * z) \overset{(2.1)}{=} (y * x) * (y * z)$$

and

$$\begin{aligned}(x * y) * ((y * z) * (x * z)) &\overset{(BE4)}{=} (y * z) * ((x * y) * (x * z)) \\ &\overset{(2.1)}{=} (y * z) * (x * (y * z)) \\ &\overset{(BE4)}{=} x * ((y * z) * (y * z)) \\ &\overset{(BE1)}{=} x * 1 \\ &\overset{(BE3)}{=} 1.\end{aligned}$$

Hence $(X, *, 1)$ is a JNB-algebra. □

From the perspective of Theorem 3.4, we can consider the JNB-algebra as a generalization of the self-distributive BE-algebra.

We can see that the JNB-algebra and the dual BCK-algebra are independent of each other by the following example.

Example 3.5. (i) Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then $(X, *, 1)$ is a dual BCK-algebra but not a JNB-algebra since

$$a * (a * b) = a * a = 1 \neq a = 1 * a = (a * a) * (a * b).$$

(ii) The JNB-algebra in Example 3.3(ii) is not a dual BCK-algebra since $c * d = 1$ and $d * c = 1$ but $c \neq d$, i.e., (DBCK4) is not valid.

We can observe that the JNB-algebra and the GE-algebra are independent of each other by the following example.

Example 3.6. (i) Let $X = \{1, a, b, c, d, e\}$ be a set with the following table.

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	d	d	e
b	1	a	1	e	d	e
c	1	a	1	1	1	1
d	1	a	1	1	1	1
e	1	a	1	1	1	1

Then $(X, *, 1)$ is a JNB-algebra but not a GE-algebra since

$$a * (b * c) = a * e = e \neq d = a * d = a * (b * d) = a * (b * (a * c)).$$

(ii) Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	1	1	c
c	1	a	a	1

Then $(X, *, 1)$ is a GE-algebra but not a JNB-algebra since

$$a * (b * c) = a * c = 1 \neq c = 1 * c = (b * a) * (b * c).$$

4. Properties of JNB-algebras

In what follows, let $(X, *, 1)$ denote a JNB-algebra unless otherwise specified.

Proposition 4.1. *Every JNB-algebra $(X, *, 1)$ satisfies the following assertions.*

(a1) $x * ((x * y) * y) = 1$,
(a2) $x * x = 1$,
(a3) $x * (x * y) = x * y$,
(a4) $x * (y * x) = 1$,
(a5) $x * 1 = 1$,
(a6) $x * y = 1$ *implies that* $(z * x) * (z * y) = 1$,
(a7) $((x * y) * z) * (y * (x * z)) = 1$,
(a8) $x * (y * (z * x)) = 1$,
(a9) $(y * z) * (y * (x * z)) = 1$,
(a10) $(y * z) * ((x * y) * (x * z)) = 1$,
(a11) $(x * y) * (y * z) = y * z$,
(a12) $x * (y * ((x * z) * z)) = 1$,

(a13) $x * (((y * x) * z) * z) = 1$,
(a14) $x * ((x * y) * (z * y)) = 1$,
(a15) $(x * (y * z)) * (y * (x * z)) = 1$,
(a16) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
(a17) $x * y = 1$ *and* $y * z = 1$ *imply* $x * z = 1$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$. If we put $x = 1$, $y = x$ and $z = y$ in (3.3) and use (3.1), then $1 = (1 * x) * ((x * y) * (1 * y)) = x * ((x * y) * y)$. Hence (a1) is valid. Also,

$$x * x \overset{(3.1)}{=} 1 * (x * x) \overset{(3.1)}{=} 1 * ((1 * x) * x) \overset{(a1)}{=} 1,$$

and so (a2) is valid.

(a3) We have $x * (x * y) \overset{(3.2)}{=} (x * x) * (x * y) \overset{(a2)}{=} 1 * (x * y) \overset{(3.1)}{=} x * y$.

(a4) We get $x * (y * x) \overset{(3.2)}{=} (y * x) * (y * x) \overset{(a2)}{=} 1$.

(a5) We have $x * 1 \overset{(a2)}{=} x * (x * x) \overset{(a3)}{=} x * x \overset{(a2)}{=} 1$.

(a6) If $x * y = 1$, then

$$\begin{aligned}(z * x) * (z * y) &\overset{(3.1)}{=} (z * x) * (1 * (z * y)) \\ &= (z * x) * ((x * y) * (z * y)) \\ &\overset{(3.3)}{=} 1.\end{aligned}$$

(a7) We have

$$\begin{aligned}((x * y) * z) * (y * (x * z)) &\overset{(3.2)}{=} ((x * y) * z) * ((x * y) * (x * z)) \\ &\overset{(3.1)}{=} ((x * y) * z) * (1 * ((x * y) * (x * z))) \\ &\overset{(a4)}{=} ((x * y) * z) * ((z * (x * z)) * ((x * y) * (x * z))) \\ &\overset{(3.3)}{=} 1.\end{aligned}$$

(a8) We obtain

$$\begin{aligned}x * (y * (z * x)) &\overset{(3.2)}{=} x * ((z * y) * (z * x)) \\ &\overset{(3.2)}{=} ((z * y) * x) * ((z * y) * (z * x)) \\ &\overset{(3.1)}{=} ((z * y) * x) * (1 * ((z * y) * (z * x)))\end{aligned}$$

$$\overset{(a4)}{=} ((z*y)*x)*((x*(z*x))*((z*y)*(z*x)))$$
$$\overset{(3.3)}{=} 1.$$

(a9) We get $(y*z)*(y*(x*z)) \overset{(3.2)}{=} z*(y*(x*z)) \overset{(a8)}{=} 1.$

(a10) We obtain

$$\begin{aligned}(y*z)*((x*y)*(x*z)) &\overset{(3.2)}{=} (y*z)*(y*(x*z))\\ &\overset{(3.2)}{=} z*(y*(x*z))\\ &\overset{(a8)}{=} 1.\end{aligned}$$

(a11) We get

$$(x*y)*(y*z) \overset{(3.2)}{=} (y*(x*y))*(y*z) \overset{(a4)}{=} 1*(y*z) \overset{(3.1)}{=} y*z.$$

(a12) We have

$$\begin{aligned}x*(y*((x*z)*z)) &\overset{(3.2)}{=} (y*x)*(y*((x*z)*z))\\ &\overset{(3.1)}{=} (y*x)*(1*(y*((x*z)*z)))\\ &\overset{(a1)}{=} (y*x)*((x*((x*z)*z))*(y*((x*z)*z)))\\ &\overset{(3.3)}{=} 1.\end{aligned}$$

(a13) We obtain

$$\begin{aligned}&x*(((y*x)*z)*z)\\ &\overset{(3.1)}{=} 1*(x*(((y*x)*z)*z))\\ &\overset{(a1)}{=} ((y*x)*(((y*x)*z)*z))*(x*(((y*x)*z)*z))\\ &\underset{(3.1)}{\overset{(a4)\&}{=}} (x*(y*x))*(((y*x)*(((y*x)*z)*z))*(x*(((y*x)*z)*z)))\\ &\overset{(3.3)}{=} 1.\end{aligned}$$

(a14) We have

$$\begin{aligned}&x*((x*y)*(z*y))\\ &\overset{(3.1)}{=} 1*(x*((x*y)*(z*y)))\\ &\overset{(3.3)}{=} ((z*x)*((x*y)*(z*y)))*(x*((x*y)*(z*y)))\end{aligned}$$

$$\overset{(3.1)}{=} 1*(((z*x)*((x*y)*(z*y)))*(x*((x*y)*(z*y))))$$
$$\overset{(a4)}{=} (x*(z*x))*(((z*x)*((x*y)*(z*y)))*(x*((x*y)*(z*y))))$$
$$\overset{(3.3)}{=} 1.$$

(a15) We have

$$(x*(y*z))*(y*(x*z))$$
$$\overset{(3.2)}{=} (x*(y*z))*((x*y)*(x*z))$$
$$\overset{(3.1)}{=} (x*(y*z))*(1*((x*y)*(x*z)))$$
$$\overset{(3.3)}{=} (x*(y*z))*(((x*y)*((y*z)*(x*z)))*((x*y)*(x*z)))$$
$$\overset{(3.2)}{=} (x*(y*z))*(((y*z)*(x*z))*((x*y)*(x*z)))$$
$$\overset{(3.2)}{=} (x*(y*z))*(((x*(y*z))*(x*z))*((x*y)*(x*z)))$$
$$\overset{(a14)}{=} 1.$$

(a16) We get

$$(x*(y*z))*((x*y)*(x*z)) \overset{(3.2)}{=} (x*(y*z))*(y*(x*z)) \overset{(a15)}{=} 1.$$

(a17) Assume that $x*y=1$ and $y*z=1$. Then

$$\begin{aligned} x*z &\overset{(3.1)}{=} 1*(x*z) = (y*z)*(x*z) \\ &\overset{(3.1)}{=} 1*((y*z)*(x*z)) \\ &= (x*y)*((y*z)*(x*z)) \\ &\overset{(3.3)}{=} 1. \end{aligned}$$

The proof has been completed. □

We can observe that every JNB-algebra $(X,*,1)$ satisfies the three conditions (BE1), (BE2) and (BE3) in the definition of BE-algebra (see (3.1), (a2) and (a5)). So, if a JNB-algebra $(X,*,1)$ satisfies the condition (BE4), then it is a BE-algebra. However, we can observe that a JNB-algebra $(X,*,1)$ does not satisfy the condition (BE4). For example, the JNB-algebra in Example 3.3(ii) does not satisfy the condition (BE4) since $a*(b*d)=c\neq d=b*(a*d)$.

In general, a JNB-algebra $(X,*,1)$ does not satisfy the condition (2.1). For example, the JNB-algebra in Example 3.3(ii) does not satisfy the condition

(2.1) since

$$(a * b) * (a * c) = d \neq c = a * d = a * (b * c).$$

Theorem 4.2. *If a JNB-algebra* $(X, *, 1)$ *satisfies the condition* (2.1)*, then it is a self-distributive BE-algebra.*

Proof. Let $(X, *, 1)$ be a JNB-algebra that satisfies the condition (2.1). It is sufficient to show that $(X, *, 1)$ satisfies the condition (BE4) because other three conditions are already checked (see (3.1), (a2) and (a5)). For every $x, y, z \in X$, we have

$$x * (y * z) \overset{(2.1)}{=} (x * y) * (x * z) \overset{(3.2)}{=} y * (x * z),$$

that is, (BE4) is valid. Hence $(X, *, 1)$ is a self-distributive BE-algebra. □

We define a binary relation "$\leq$" on a JNB-algebra $(X, *, 1)$ as follows:

$$(\forall x, y \in X)(x \leq y \text{ if and only if } x * y = 1). \tag{4.1}$$

It is clear that the relation $\leq$ is reflexive and transitive by (a2) and (a17), respectively, that is, $\leq$ is a quasi order. But it is not a partial order because it is not antisymmetric as seen in the following example.

Example 4.3. Let $X = \{1, a, b, c, d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	1
b	1	1	1	c	1
c	1	b	b	1	d
d	1	a	a	c	1

Then $(X, *, 1)$ is a JNB-algebra. Since $a * b = 1$ and $b * a = 1$ but $a \neq b$, it is not antisymmetric.

For every $x, y \in X$, we define two sets:

$$\overrightarrow{x} = \{z \in X \mid x * z = 1\} \text{ and } \overrightarrow{(x, y)} = \{z \in X \mid x * (y * z) = 1\},$$

called the x-*upper set* and the (x, y)-*upper set*, respectively. We can observe that $1, x \in \overrightarrow{x}$ and $1, x, y \in \overrightarrow{(x, y)}$. Especially, $\overrightarrow{1} = \{1\} = \overrightarrow{(1, 1)}$. In general, $\overrightarrow{x}$ and $\overrightarrow{(x, y)}$ cannot be identical. In fact, $\overrightarrow{a} = \{1, a, b, d\}$ and $\overrightarrow{(a, c)} = X$ in Example 4.3.

We explore the conditions under which the x-upper set and the (x, y)-upper set can be identical for $x, y \in X$.

Proposition 4.4. *For every $x, y \in X$, we have*

$$y \in \overrightarrow{x} \Leftrightarrow \overrightarrow{x} = \overrightarrow{(x, y)}. \tag{4.2}$$

Proof. Assume that $y \in \overrightarrow{x}$. Then $x * y = 1$. Let $a \in \overrightarrow{x}$. Then $x * a = 1$. That implies $y * (x * a) = (x * y) * (x * a) = 1$. Therefore $x * (y * a) = 1$. So that $a \in \overrightarrow{(x, y)}$. Thus $\overrightarrow{x} \subseteq \overrightarrow{(x, y)}$. Let $a \in \overrightarrow{(x, y)}$. Then $x * (y * a) = 1$. Hence $y * (x * a) = 1$. So that $(x * y) * (x * a) = 1$, which gives $x * a = 1 * (x * a) = 1$. Therefore $a \in \overrightarrow{x}$. Hence $\overrightarrow{(x, y)} \subseteq \overrightarrow{x}$. Thus $\overrightarrow{x} = \overrightarrow{(x, y)}$.

Conversely, assume that $\overrightarrow{x} = \overrightarrow{(x, y)}$. Since $x * (y * y) = 1$, we get that $y \in \overrightarrow{(x, y)}$. Therefore $y \in \overrightarrow{x}$. □

Proposition 4.5. *For every $x, y \in X$, we have*

(i) *The x-upper set can be represented by the intersection of (x, y)-upper sets for all $y \in X$, that is, $\overrightarrow{x} = \bigcap_{y \in X} \overrightarrow{(x, y)}$.*
(ii) $\overrightarrow{(x, y)} = \overrightarrow{(y, x)}$.
(iii) *$x \leq y$ if and only if $\overrightarrow{y} \subseteq \overrightarrow{x}$.*
(iv) *$x \leq y$ and $y \leq x$ if and only if $\overrightarrow{x} = \overrightarrow{y}$.*

Proof. (i). Let $z \in \overrightarrow{x}$. Then $x * z = 1$ and hence $y * (x * z) = 1$, for all $y \in X$ which implies that $1 = y * (x * z) \leq x * (y * z)$. Therefore $x * (y * z) = 1$. So that $z \in \overrightarrow{(x, y)}$, for all $y \in X$ which gives that $z \in \bigcap_{y \in X} \overrightarrow{(x, y)}$. Hence $\overrightarrow{x} \subseteq \bigcap_{y \in X} \overrightarrow{(x, y)}$. Now, $z \in \bigcap_{y \in X} \overrightarrow{(x, y)}$ implies that $x * (y * z) = 1$, for all $y \in X$. Since $1 \in X$, we have $x * (1 * z) = 1$. Hence $x * z = 1$. Therefore $z \in \overrightarrow{x}$. Thus $\bigcap_{y \in X} \overrightarrow{(x, y)} \subseteq \overrightarrow{x}$. Hence $\overrightarrow{x} = \bigcap_{y \in X} \overrightarrow{(x, y)}$.

(ii). We have

$$z \in \overrightarrow{(x, y)} \Leftrightarrow x * (y * z) = 1 \Leftrightarrow y * (x * z) = 1 \Leftrightarrow z \in \overrightarrow{(y, x)}.$$

Therefore $\overrightarrow{(x, y)} = \overrightarrow{(y, x)}$.

(iii). Assume that $x \leq y$. Let $a \in \overrightarrow{y}$. Then $y * a = 1$. That implies $y \leq a$. By our assumption, we get that $x \leq a$. Therefore $a \in \overrightarrow{x}$. Hence $\overrightarrow{y} \subseteq \overrightarrow{x}$. Conversely, assume that $\overrightarrow{y} \subseteq \overrightarrow{x}$. For any $y \in X$, we have that $y * y = 1$. Therefore $y \in \overrightarrow{y}$. Hence $y \in \overrightarrow{x}$ which gives $x * y = 1$. Therefore $x \leq y$.

(iv). It is straightforward by (iii). □

5. JNB-subalgebras and JNB-deductive systems

In this section, we introduce JNB-subalgebra and JNB-deductive system in a JNB-algebra and study their properties. We characterize JNB-deductive systems in terms of upper sets.

Definition 5.1. A subset D of X is called a *JNB-subalgebra* of $(X, *, 1)$ if $x * y \in D$ for all $x, y \in D$.

Example 5.2. Let $(X, *, 1)$ be the JNB-algebra in Example 4.3. Then the set $D := \{1, a, b\}$ is a JNB-subalgebra of $(X, *, 1)$.

Definition 5.3. A subset D of X is called a *JNB-deductive system* of $(X, *, 1)$ if it satisfies:

(ds1) $1 \in D$
(ds2) if $x * y \in D$ and $x \in D$ then $y \in D$.

Obviously, $\{1\}$ and X are JNB-deductive systems of $(X, *, 1)$. A JNB-deductive system D is said to be *proper* if $D \neq X$.

Example 5.4. Consider the JNB-algebra $(X, *, 1)$ in Example 3.3(ii). Then we can verify that $D_1 = \{1\}$, $D_2 = \{1, a\}$, $D_3 = \{1, a, b\}$, $D_4 = \{1, b, c, d\}$, and $D_5 = X$ are all JNB-deductive systems of $(X, *, 1)$.

Proposition 5.5. *Every JNB-deductive system D of $(X, *, 1)$ satisfies:*

$$(\forall \mathfrak{a}, x \in X)(\mathfrak{a} \in D \;\Rightarrow\; (\mathfrak{a} * x) * x \in D). \tag{5.1}$$

Proof. Let D be a JNB-deductive system of $(X, *, 1)$, and let $x \in X$ and $\mathfrak{a} \in D$. Then $\mathfrak{a} * ((\mathfrak{a} * x) * x) = 1 \in D$ by (a1) and (ds1). Hence $(\mathfrak{a} * x) * x \in D$ by (ds2). □

Theorem 5.6. *A subset D of X is a JNB-deductive system of $(X, *, 1)$ if and only if it satisfies:*

$$(\forall x, \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a}, \mathfrak{b} \in D,\ \mathfrak{a} * (\mathfrak{b} * x) = 1 \;\Rightarrow\; x \in D). \tag{5.2}$$

Proof. Suppose D is a JNB-deductive system of X. Let $x, \mathfrak{a}, \mathfrak{b} \in X$ be such that $\mathfrak{a}, \mathfrak{b} \in D$ and $\mathfrak{a} * (\mathfrak{b} * x) = 1$. Then $\mathfrak{a} * (\mathfrak{b} * x) \in D$ by (ds1), and so $x \in D$ by (ds2).

Conversely, assume that D satisfies (5.2). Since $\mathfrak{a} * (\mathfrak{a} * 1) = 1$ for all $\mathfrak{a} \in D$, we have $1 \in D$ by (5.2). Let $x, y \in X$ be such that $x * y \in D$ and $x \in D$. Then, since $x * ((x * y) * y) \overset{(a1)}{=} 1 \in D$, we have $y \in D$ by (5.2). Hence D is a JNB-deductive system of $(X, *, 1)$. □

Theorem 5.7. *A subset D of X is a JNB-deductive system of $(X,*,1)$ if and only if $\overrightarrow{(x,y)} \subseteq D$ for all $x,y \in D$.*

Proof. Assume that D is a JNB-deductive system of $(X,*,1)$. For every $x,y \in D$, if $z \in \overrightarrow{(x,y)}$, then $x*(y*z) = 1 \in D$ and so $z \in D$. Hence $\overrightarrow{(x,y)} \subseteq D$.

Conversely, assume that $\overrightarrow{(x,y)} \subseteq D$ for all $x,y \in D$. For every $z \in D$, we have $1 \in \overrightarrow{(z,z)} \subseteq D$. Let $x,y \in X$ be such that $x*y \in D$ and $x \in D$. Then $y \in \overrightarrow{(x*y,x)} \subseteq D$. Therefore D is a JNB-deductive system of $(X,*,1)$. □

Proposition 5.8. *Let D be a JNB-deductive system of $(X,*,1)$. Then D contains the x-upper set for all $x \in D$, and D can be represented by the union of (x,y)-upper sets for all $x,y \in D$.*

Proof. For every $x \in D$, if $z \in \overrightarrow{x}$, then $x*z = 1 \in D$ and hence $z \in D$. Therefore $\overrightarrow{x} \subseteq D$.

Let $x,y \in D$. It is clear that $x \in \overrightarrow{(x,1)}$. Thus

$$D \subseteq \bigcup_{x\in D} \overrightarrow{(x,1)} \subseteq \bigcup_{x,y\in D} \overrightarrow{(x,y)}.$$

If $z \in \bigcup_{x,y\in D} \overrightarrow{(x,y)}$, then $z \in \overrightarrow{(\mathfrak{a},\mathfrak{b})}$ for some $\mathfrak{a},\mathfrak{b} \in D$, and so $\mathfrak{a}*(\mathfrak{b}*z) = 1 \in D$. It follows from (ds2) that $z \in D$. Thus $\bigcup_{x,y\in D} \overrightarrow{(x,y)} \subseteq D$, and consequently $D = \bigcup_{x,y\in D} \overrightarrow{(x,y)}$. □

The proof of the following lemma is straightforward and hence we omit the proof.

Lemma 5.9. *If $\{D_i\}_{i\in\Lambda}$ is a family of JNB-deductive systems (resp., JNB-subalgebras) of $(X,*,1)$, then so is $\bigcap_{i\in\Lambda} D_i$.*

The following example shows that the union of JNB-deductive systems (resp., JNB-subalgebras) of $(X,*,1)$ may not be a JNB-deductive system (resp., JNB-subalgebra) of $(X,*,1)$.

Example 5.10. Let $X = \{1,a,b,c,d,e,f\}$ be a set with the following table.

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	e	d	e	e
b	1	1	1	d	d	e	d
c	1	1	1	1	1	1	1
d	1	a	b	a	1	1	a
e	1	a	b	a	1	1	a
f	1	1	1	1	1	1	1

Then $(X, *, 1)$ is a JNB-algebra. Let $D = \{1, a, b\}$ and $G = \{1, d, e\}$. Then D and G are JNB-deductive systems of $(X, *, 1)$. But $D \cup G = \{1, a, b, d, e\}$ is not a JNB-deductive system of $(X, *, 1)$ since $b * f = d \in D \cup G$ but $f \notin D \cup G$.

We denote the set of all JNB-deductive systems of $(X, *, 1)$ by $\mathcal{D}(X)$. Since the set $\mathcal{D}(X)$ is closed under arbitrary intersections, we have the following theorem.

Theorem 5.11. $(\mathcal{D}(X), \subseteq)$ *is a complete lattice.*

We now establish the relationship between a JNB-subalgebra and a JNB-deductive system.

Theorem 5.12. *Every JNB-deductive system is a JNB-subalgebra.*

Proof. Let D be a JNB-deductive system of $(X, *, 1)$. If $x, y \in D$, then

$$y * (x * y) \overset{(a4)}{=} 1 \in D,$$

and so $x*y \in D$ by (ds2). Hence $(X, *, 1)$ is a JNB-subalgebra of $(X, *, 1)$. □

In Example 4.3, we can observe that $D := \{1, a, b\}$ is a JNB-subalgebra of $(X, *, 1)$. But it is not a JNB-deductive system of $(X, *, 1)$ since $a * d = 1$ and $a \in D$ but $d \notin D$. Therefore, we know the converse of Theorem 5.12 does not hold in general.

Lemma 5.13. *Every JNB-algebra* $(X, *, 1)$ *satisfies:*

$$(\forall x \in X)(1 \leq x \;\Rightarrow\; x = 1).$$

Proof. Straightforward. □

Theorem 5.14. *For every* $x, y \in X$*, the* (x, y)*-upper set* $\overrightarrow{(x, y)}$ *is both a JNB-subalgebra and a JNB-deductive system of* $(X, *, 1)$*.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \overrightarrow{(x, y)}$. Then $x * (y * \mathfrak{a}) = 1$ and $x * (y * \mathfrak{b}) = 1$, that is, $x \leq y * \mathfrak{a}$ and $x \leq y * \mathfrak{b}$. Using (a6) and (a15) in Proposition 4.1, we get $\mathfrak{a} * x \leq \mathfrak{a} * (y * \mathfrak{b}) \leq y * (\mathfrak{a} * \mathfrak{b})$. Since $\leq$ is transitive, it follows from (a4) and (a6) that

$$1 = x * (\mathfrak{a} * x) \leq x * (y * (\mathfrak{a} * \mathfrak{b}))\,.$$

Hence $x * (y * (\mathfrak{a} * \mathfrak{b})) = 1$ by Lemma 5.13, that is, $\mathfrak{a} * \mathfrak{b} \in \overrightarrow{(x, y)}$. Thus $\overrightarrow{(x, y)}$ is a JNB-subalgebra of $(X, *, 1)$. Let $x, y \in X$. Since $x * (y * 1) = x * 1 = 1$ by (a5), we have $1 \in \overrightarrow{(x, y)}$. Let $\mathfrak{a}, \mathfrak{b} \in X$ be such that $\mathfrak{a} * \mathfrak{b} \in \overrightarrow{(x, y)}$ and $\mathfrak{a} \in \overrightarrow{(x, y)}$. Then $x * (y * (\mathfrak{a} * \mathfrak{b})) = 1$ and $x * (y * \mathfrak{a}) = 1$. Hence

$$x \leq y * (\mathfrak{a} * \mathfrak{b}) \leq \mathfrak{a} * (y * \mathfrak{b}) = (y * \mathfrak{a}) * (y * \mathfrak{b})$$

by (4.1), (a15) and (3.2), and so $x \leq (y * \mathfrak{a}) * (y * \mathfrak{b})$ since $\leq$ is transitive. Therefore $1 = x * ((y * \mathfrak{a}) * (y * \mathfrak{b})) \leq (y * \mathfrak{a}) * (x * (y * \mathfrak{b}))$, which implies from Lemma 5.13 that $(y * \mathfrak{a}) * (x * (y * \mathfrak{b})) = 1$, i.e., $y * \mathfrak{a} \leq x * (y * \mathfrak{b})$. Hence $1 = x * (y * \mathfrak{a}) \leq x * (x * (y * \mathfrak{b})) = x * (y * \mathfrak{b})$ by (a6) and (a3), and so $x * (y * \mathfrak{b}) = 1$ by Lemma 5.13. Thus $\mathfrak{b} \in \overrightarrow{(x, y)}$. Therefore $\overrightarrow{(x, y)}$ is a JNB-deductive system of $(X, *, 1)$. □

The combination of Proposition 4.5(i), Lemma 5.9 and Theorem 5.14 derives to the following corollary.

Corollary 5.15. *For every $x \in X$, the x-upper set $\overrightarrow{x}$ is both a JNB-subalgebra and a JNB-deductive system of $(X, *, 1)$.*

Proposition 5.16. *Every JNB-deductive system D of $(X, *, 1)$ satisfies:*

$$(\forall \mathfrak{a}, x \in X)(\mathfrak{a} \in D,\ \mathfrak{a} \leq x \ \Rightarrow \ x \in D). \tag{5.3}$$

Proof. Let $\mathfrak{a}, x \in X$ be such that $\mathfrak{a} \in D$ and $\mathfrak{a} \leq x$. Then $\mathfrak{a} * x = 1 \in D$ by (ds1), and so $x \in D$ by (ds2). □

Definition 5.17. A nonempty subset F of X is called a *JNB-filter* of $(X, *, 1)$ if it satisfies:

$$(\forall x, y \in X)\ (y \in F \ \Rightarrow \ x * y \in F), \tag{5.4}$$

$$(\forall x, \mathfrak{a}, \mathfrak{b} \in X)\ (\mathfrak{a}, \mathfrak{b} \in F \ \Rightarrow \ (\mathfrak{a} * (\mathfrak{b} * x)) * x \in F). \tag{5.5}$$

Example 5.18. Consider the JNB-algebra $(X, *, 1)$ in Example 3.3(ii). It is routine to verify that $F := \{1, a, b\}$ is a JNB-filter of $(X, *, 1)$

Lemma 5.19. *Let F be a JNB-filter of $(X, *, 1)$. Then*

(b1) $1 \in F$,
(b2) $(\forall x, y \in X)$ $(x \in F \;\Rightarrow\; (x * y) * y \in F)$,
(b3) $(\forall \mathfrak{a}, x \in X)$ $(\mathfrak{a} \in F,\, \mathfrak{a} \leq x \;\Rightarrow\; x \in F)$,
(b4) $(\forall x, y, \mathfrak{b} \in X)$ $(x * (\mathfrak{b} * y) \in F,\, \mathfrak{b} \in F \;\Rightarrow\; x * y \in F)$.

Proof. Let F be a JNB-filter of $(X, *, 1)$. Then there exists $x \in F$, and so $1 = x * x \in F$ by (a2) and (5.4). Thus (b1) is valid. Let $x \in F$ and $y \in X$. Then $(x * y) * y = 1 * ((x * y) * y) \in F$ by (3.1), (b1) and (5.5). Thus (b2) is valid. Let $\mathfrak{a} \in F, x \in X$ and $\mathfrak{a} \leq x$. Then $\mathfrak{a} * x = 1$, and so $x = 1 * x = (\mathfrak{a} * x) * x \in F$ by (3.1) and (b2). Thus (b3) is valid. Let $x, y, \mathfrak{b} \in X$ be such that $x*(\mathfrak{b}*y) \in F$ and $\mathfrak{b} \in F$. Then $x*(\mathfrak{b}*y) \leq \mathfrak{b}*(x*y)$ by (a15), which implies from (b3) that $\mathfrak{b} * (x * y) \in F$. Using (3.1), (a2) and (5.5), we have

$$x * y = 1 * (x * y) = ((\mathfrak{b} * (x * y)) * (\mathfrak{b} * (x * y))) * (x * y) \in F$$

which shows that (b4) is valid. □

Lemma 5.20. *If a nonempty subset F of X satisfies two conditions* (b1) *and* (b4), *then*

$$(\forall \mathfrak{b}, x \in X)(\mathfrak{b} \in F,\, \mathfrak{b} \leq x \;\Rightarrow\; x \in F). \tag{5.6}$$

Proof. Assume that F satisfies (b1) and (b4). Let $\mathfrak{b}, x \in X$ be such that $\mathfrak{b} \in F$ and $\mathfrak{b} \leq x$. Then $1 * (\mathfrak{b} * x) = \mathfrak{b} * x = 1 \in F$ by (3.1) and (b1). It follows from (3.1) and (b4) that $x = 1 * x \in F$. □

Theorem 5.21. *If a nonempty subset F of X satisfies two conditions* (b1) *and* (b4), *then it is a JNB-filter of* $(X, *, 1)$.

Proof. Assume that F satisfies (b1) and (b4). Let $x \in X$ and $y \in F$. Then $x * (y * y) = x * 1 = 1 \in F$ by (a2), (a5) and (b1), and so $x * y \in F$ by (b4). Let $x \in X$ and $\mathfrak{a}, \mathfrak{b} \in F$. Then $(\mathfrak{a} * x) * (\mathfrak{a} * x) = 1 \in F$ by (a2) and (b1), which implies from (b4) that $(\mathfrak{a} * x) * x \in F$. Using (a10) induces

$$((\mathfrak{a} * x) * x) * ((\mathfrak{b} * (\mathfrak{a} * x)) * (\mathfrak{b} * x)) = 1,$$

that is, $(\mathfrak{a} * x) * x \leq (\mathfrak{b} * (\mathfrak{a} * x)) * (\mathfrak{b} * x)$. Hence $(\mathfrak{b} * (\mathfrak{a} * x)) * (\mathfrak{b} * x) \in F$ by Lemma 5.20, and thus $(\mathfrak{b} * (\mathfrak{a} * x)) * x \in F$ by (b4). Therefore F is a JNB-filter of $(X, *, 1)$. □

Let F be a JNB-filter of $(X, *, 1)$. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Then

$$y \overset{(3.1)}{=} 1 * y \overset{(a2)}{=} ((x * y) * (x * y)) * y \overset{(5.5)}{\in} F$$

which shows that F satisfies the condition (ds2). Hence we have the following theorem.

Theorem 5.22. *Every JNB-filter is a JNB-deductive system.*

Corollary 5.23. *If a nonempty subset F of X satisfies two conditions* (b1) *and* (b4)*, then it is a JNB-deductive system of* $(X, *, 1)$*.*

We discuss the converse of Theorem 5.22.

Theorem 5.24. *Every JNB-deductive system is a JNB-filter.*

Proof. Let F be a JNB-deductive system of $(X, *, 1)$. If $y \in F$, then $y * (x * y) = 1 \in F$ by (a4) and (ds1), and hence $x * y \in F$ by (ds2) for all $x \in F$. Let $\mathfrak{a}, \mathfrak{b} \in F$. Since

$$\mathfrak{a} * ((\mathfrak{a} * (\mathfrak{b} * x)) * (\mathfrak{b} * x)) = 1 \in F$$

by (a1) and (ds1), we have $(\mathfrak{a} * (\mathfrak{b} * x)) * (\mathfrak{b} * x) \in F$ by (ds2). Also, since

$$(\mathfrak{a} * (\mathfrak{b} * x)) * (\mathfrak{b} * x) \leq \mathfrak{b} * ((\mathfrak{a} * (\mathfrak{b} * x)) * x)$$

by (a15), it follows from Proposition 5.16 that $\mathfrak{b} * ((\mathfrak{a} * (\mathfrak{b} * x)) * x) \in F$. Thus $(\mathfrak{a} * (\mathfrak{b} * x)) * x \in F$ by (ds2). Therefore F is a JNB-filter of $(X, *, 1)$. □

Remark 5.25. Through Theorems 5.22 and 5.24, we can see that the JNB-deductive system and the JNB-filter are coincident concepts.

We finally discuss a characterization of the JNB-filter.

Theorem 5.26. *A subset F of X is a JNB-filter of* $(X, *, 1)$ *if and only if it satisfies* (ds1) *and*

$$(\forall x, \mathfrak{a}, y \in X)(x * \mathfrak{a} \in F,\ \mathfrak{a} * y \in F \ \Rightarrow\ x * y \in F). \tag{5.7}$$

Proof. Suppose that F is a JNB-filter of $(X, *, 1)$. Then F is a JNB-deductive system of $(X, *, 1)$. Hence (ds1) is valid. Let $x, \mathfrak{a}, y \in X$ be such that $x * \mathfrak{a} \in F$ and $\mathfrak{a} * y \in F$. Then $(\mathfrak{a} * y) * ((x * \mathfrak{a}) * (x * y)) = 1 \in F$ by (a10) and (ds1). It follows from (ds2) that $x * y \in F$. Hence (5.7) is valid.

Assume that F satisfies (ds1) and (5.7). Let $x, y \in X$ be such that $x \in F$ and $x * y \in F$. Then $1 * x = x \in F$ by (3.1), and so $y = 1 * y \in F$ by (3.1) and (5.7). Hence F is a JNB-deductive system, and so a JNB-filter of $(X, *, 1)$. □

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JNB–ALGEBRAS

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JNB–جبرها

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به عنوان تعمیمی از BE–جبر خود–توزیعی، JNB–جبر معرفی شده و خواص اساسی آن مورد بررسی قرار گرفته است. این جبر می‌تواند نقش‌های مختلفی در مطالعه جبر منطقی، از جمله BCK–جبر ایفا کند. ابتدا، مثال‌هایی ارائه شده که نشان می‌دهند سه اصل JNB–جبر مستقل از یکدیگر هستند. خواص اساسی JNB–جبرها که برای مطالعه نظریه‌های مختلف درباره JNB–جبرها مورد نیاز هستند، مورد بررسی قرار گرفته است. مجموعه‌های بالایی بر اساس یک و دو عنصر معرفی شده و ویژگی‌های مرتبط با آن‌ها بررسی شده‌اند. دو مفهوم به نام سیستم JNB–قیاسی و JNB–فیلتر معرفی شده و خواص آن‌ها مورد تحقیق قرار گرفته‌اند. ویژگی‌های سیستم JNB–قیاسی و JNB–فیلتر مورد بحث قرار گرفته‌اند. در نهایت نشان داده شده است که سیستم JNB–قیاسی و JNB–فیلتر مطابقت دارند.

کلمات کلیدی: JNB–جبر، JNB–زیرجبر، مجموعه بالایی، سیستم JNB–قیاسی، JNB–فیلتر.