Journal of Algebraic Systems, Vol. 13(No. 3): (2025), pp 137-155.
https://doi.org/10.22044/JAS.2024.13825.1775

DEDEKIND-MACNEILLE COMPLETION OF THE ROUGH SETS SYSTEM AS PASTING OF ROUGH APPROXIMATION LATTICES

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Abstract. The pattern of embedding of rough approximation lattices defined by a reflexive relation in its Dedekind-MacNeille completion of rough sets system is taken up for study in this work. The reflexive relation R for which the Dedekind-MacNeille completion of rough sets system defined by R is the pasting of its rough approximation lattices is characterized. Some properties of the Dedekind-MacNeille completion of rough sets system defined by a reflexive relation R are also discussed.

1. Introduction

Pawlak [15] proposed a rough set theory to deal with uncertainty caused by lack of information. The theory starts from an approximation space (U, E), where E is an equivalence relation that represents the indiscernibility of objects in U, with the available information. This equivalence relation was relaxed later and the rough set was defined from the generalized approximation space (U, R), where R is the information relation derived from different forms of information system [13, 19]. The set of all rough sets defined from an approximation space is denoted by $\mathcal{R}^*$ and is called as rough sets system. The rough sets are represented in the form of an approximation pair $(A^{\blacktriangledown}, A^{\blacktriangle})$ for $A \subseteq U$ to study its algebraic structures. The system is ordered coordinate-wise as $(A^{\blacktriangledown}, A^{\blacktriangle}) \leq (B^{\blacktriangledown}, B^{\blacktriangle}) \Leftrightarrow A^{\blacktriangledown} \subseteq B^{\blacktriangledown}$ and $A^{\blacktriangle} \subseteq B^{\blacktriangle}$. The rough sets system together with the ordering $\leq$, forms a partially ordered set $(\mathcal{R}^*, \leq)$ which turns out into lattice-based algebraic structures for some reflexive-based relations.

The foundations of the algebraic structures of $(\mathcal{R}^*, \leq)$ were initially based on an equivalence relation [2, 7, 14, 16]. Later, the ordered structure of $(\mathcal{R}^*, \leq)$ for various relations like quasi-order relation (reflexive and transitive) [12], tolerance relation (reflexive and symmetric) [10] and simply reflexive relation [9] were studied.

Published online: 9 April 2024
MSC(2020): Primary: 06A06; Secondary: 06B23, 18B15.
Keywords: Rough Sets System; Rough approximations; Join-dense; Pasting of lattices.
Received: 11 November 2023, Accepted: 9 February 2024.

Järvinen in [9] has given in general a few completions for $(\mathcal{R}^*, \leq)$ defined by a binary relation R that is not a lattice. He has also proposed a problem of finding the smallest completion of the $(\mathcal{R}^*, \leq)$ as defined by a binary relation. The author of the current paper has given a solution to this problem by finding the Dedekind-MacNeille completion of $(\mathcal{R}^*, \leq)$ defined by binary relation and has also studied the algebraic structures of this completion in [18]. However, the reflexive condition on binary relation is considered as necessary for many applications. Hence the rough sets defined by reflexive relation are studied in this paper. Here, reflexive relation R is stated in the sense that R can be any arbitrary binary relation with the least property of it to be reflexive.

In this paper, first, it is shown that the lattice structure of rough approximations as defined by a reflexive relation R is order embedded in the Dedekind-MacNeille completion of $(\mathcal{R}^*, \leq)$ defined by R. Next, it is shown that the pasting of rough approximation lattices over the common Boolean sublattice is order embedded in its Dedekind-MacNeille completion of $(\mathcal{R}^*, \leq)$. It will be of great interest to study the lattice structure of a rough set, if it can be obtained by just pasting its rough approximation lattices. So, with this concept in mind, the reflexive relation R is characterized for which the Dedekind-MacNeille completion of $(\mathcal{R}^*, \leq)$ is obtained by simply pasting its rough approximation lattices over the common Boolean sublattice. Further, some of the algebraic properties of the Dedekind-MacNeille completion of $(\mathcal{R}^*, \leq)$ defined by a reflexive relation is analyzed.

2. Properties of Rough Approximations

In an approximation space (U, R) with R as a reflexive relation, the lower and upper (rough) approximations of a subset A of U are defined as

$$\begin{aligned} A^{\blacktriangledown} &= \{x \in U | R(x) \subseteq A\} \\ A^{\blacktriangle} &= \{x \in U | R(x) \cap A \neq \emptyset\} \end{aligned} \tag{2.1}$$

where $R(x) = \{y \in U | xRy\}$.

$$\wp(U)^{\blacktriangledown} = \{A^{\blacktriangledown} | A \subseteq U\} \text{ and } \wp(U)^{\blacktriangle} = \{A^{\blacktriangle} | A \subseteq U\}.$$

If the rough approximations of two subsets of U are equal, then it can be said that one is roughly equal to the other and is denoted by $\equiv$. This rough equality $(\equiv)$ relation is an equivalence relation on $\wp(U)$, and the resulting equivalence classes are called rough sets. The family of all rough sets of an approximation space (U, R) is called as rough sets system defined by R and is denoted by $\mathcal{R}^*$. That is,

$$\mathcal{R}^* = \{(Y^\blacktriangledown, Y^\blacktriangle) | Y \subseteq U\}$$

$\mathcal{R}^*$ is ordered canonically by coordinate-wise order as

$$(X^\blacktriangledown, X^\blacktriangle) \leq (Y^\blacktriangledown, Y^\blacktriangle) \Leftrightarrow X^\blacktriangledown \subseteq Y^\blacktriangledown \, and \, X^\blacktriangle \subseteq Y^\blacktriangle \tag{2.2}$$

Let us denote $R^{-1}(x) = \{y \in U | yRx\}$ and the rough approximations of $A \subseteq U$ with respect to inverse relation R^{-1} are defined as follows:

$$\begin{aligned} A^\triangledown &= \{x \in U | R^{-1}(x) \subseteq A\} \\ A^\vartriangle &= \{x \in U | R^{-1}(x) \cap A \neq \emptyset\} \end{aligned} \tag{2.3}$$

Järvinen[8] showed that the pair of mappings $(^\blacktriangle, ^\triangledown)$ and $(^\vartriangle, ^\blacktriangledown)$ are Galois connections on $\wp(U)$. As a consequence of this, the following hold for all $Z \subseteq U$:

$$Z^{\blacktriangledown\vartriangle} \subseteq Z \subseteq Z^{\vartriangle\blacktriangledown}; Z^{\triangledown\blacktriangle} \subseteq Z \subseteq Z^{\blacktriangle\triangledown}$$

$$Z^\blacktriangle = Z^{\blacktriangle\triangledown\blacktriangle}, \; Z^\triangledown = Z^{\triangledown\blacktriangle\triangledown}, \; Z^\vartriangle = Z^{\vartriangle\blacktriangledown\vartriangle} \text{ and } Z^\blacktriangledown = Z^{\blacktriangledown\vartriangle\blacktriangledown}$$

$$Z^\vartriangle = \bigcup_{y \in Z} \{y\}^\vartriangle = \bigcup_{y \in Z} R(y) \text{ and } Z^\blacktriangle = \bigcup_{y \in Z} \{y\}^\blacktriangle = \bigcup_{y \in Z} R^{-1}(y)$$

And, $(\wp(U)^\vartriangle, \subseteq) \cong (\wp(U)^\blacktriangledown, \subseteq) \cong (\wp(U)^\blacktriangle, \supseteq) \cong (\wp(U)^\triangledown, \supseteq)$ also holds. The readers can refer to the literature [3, 5, 8] for basic definitions and results in lattice theory and rough set theory.

3. Some Results on Dedekind-MacNeille Completion of the Rough Sets System defined by a Reflexive relation

Generally, for every reflexive relation R, $(\mathcal{R}^*, \leq)$ is not a lattice. Järvinen has provided a few possible completions for $(\mathcal{R}^*, \leq)$ defined by a binary relation in [9] and has concluded the paper with an open problem stating that "Determine the smallest completion of $(\mathcal{R}^*, \leq)$". The author of the current paper has given the solution to the problem in [18] by finding the Dedekind-MacNeille completion of $(\mathcal{R}^*, \leq)$. The Dedekind-MacNeille completion of the poset $(\mathcal{R}^*, \leq)$ is as follows

$$\begin{aligned} [\wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle]' = \{(A, B) \in \wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle | A^{\vartriangle\blacktriangle} \subseteq B \text{ and} \\ A \cap \mathcal{B} = B \cap \mathcal{B}\} \end{aligned} \tag{3.1}$$

where $\mathcal{B} = \{x \in U | R(x) = \{x\}\}$.

The lattice operations of $[\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}]'$ are defined as follows [18] for any index set I and $\{(X_i^{\blacktriangledown}, Y_i^{\blacktriangle})\}_{i\in I} \subseteq [\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}]'$,

$$\bigwedge_{i\in I}(X_i^{\blacktriangledown}, Y_i^{\blacktriangle}) = (\underline{\triangle}_{i\in I} X_i^{\blacktriangledown}, \overline{\wedge}_{i\in I} Y_i^{\blacktriangle})$$

$$\bigvee_{i\in I}(X_i^{\blacktriangledown}, Y_i^{\blacktriangle}) = (\underline{\vee}_{i\in I} X_i^{\blacktriangledown}, \overline{\triangledown}_{i\in I} Y_i^{\blacktriangle})$$

where $\underline{\vee}$ and $\underline{\triangle}$ are the join and meet operations in $\wp(U)^{\blacktriangledown}$ respectively. Similarly, $\overline{\triangledown}$ and $\overline{\wedge}$ are the join and meet operations in $\wp(U)^{\blacktriangle}$ respectively. Also, the operator $\sim$ on $[\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}]'$, defined by $\sim (X^{\blacktriangledown}, Y^{\blacktriangle}) = (Y^{c\blacktriangledown}, X^{c\blacktriangle})$, satisfies the following properties : for $(X^{\blacktriangledown}, Y^{\blacktriangle}), (V^{\blacktriangledown}, W^{\blacktriangle}) \in [\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}]'$,

(i) $\sim [(X^{\blacktriangledown}, Y^{\blacktriangle}) \bigvee (V^{\blacktriangledown}, W^{\blacktriangle})] = \sim (X^{\blacktriangledown}, Y^{\blacktriangle}) \bigwedge \sim (V^{\blacktriangledown}, W^{\blacktriangle})$.
(ii) $\sim\sim (X^{\blacktriangledown}, Y^{\blacktriangle}) = (X^{\blacktriangledown}, Y^{\blacktriangle})$.
(iii) $(X^{\blacktriangledown}, Y^{\blacktriangle}) \bigwedge \sim (X^{\blacktriangledown}, Y^{\blacktriangle}) \leq (V^{\blacktriangledown}, W^{\blacktriangle}) \bigvee \sim (V^{\blacktriangledown}, W^{\blacktriangle})$.

Let us denote the ordered structure of this completion $[\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}]'$ by $(\mathcal{R}_c^*, \leq)$. From the above properties of the operator $\sim$, $(\mathcal{R}_c^*, \leq)$ is a symmetric lattice.

Lemma 3.1. *Let R be any reflexive relation on the set U. For all $X \subseteq U$,*

$$X^{\blacktriangledown} = \bigcup\{\{x\}^{\triangle\blacktriangledown} | \{x\}^{\triangle} \subseteq X\}.$$

Proof. For $X \subseteq U$, $x \in X^{\blacktriangledown} \Rightarrow \{x\}^{\triangle} = R(x) \subseteq X$ and always $x \in \{x\}^{\triangle\blacktriangledown}$. Let $y \in \{x\}^{\triangle\blacktriangledown}$ and $\{x\}^{\triangle} \subseteq X$, for some $x \in U$. Then,

$$R(y) \subseteq \{x\}^{\triangle} \subseteq X \Rightarrow y \in X^{\blacktriangledown}.$$

Hence $X^{\blacktriangledown} = \bigcup\{\{x\}^{\triangle\blacktriangledown} | \{x\}^{\triangle} \subseteq X\}$. □

We have $\mathcal{B}^{\blacktriangledown} = \mathcal{B}$, by the above Lemma 3.1 and by duality ($X^{\blacktriangledown c} = X^{c\blacktriangle}$), $\mathcal{B}^{c\blacktriangle} = \mathcal{B}^c$. Then for any reflexive relation R on the set U and for all $X \in \wp(U)$, the following hold:

$$(X^{\blacktriangle} \cap \mathcal{B})^{\blacktriangledown} = X^{\blacktriangle\blacktriangledown} \cap \mathcal{B} = X^{\blacktriangle} \cap \mathcal{B} \tag{3.2}$$

$$(X^{\blacktriangledown} \cup \mathcal{B}^c)^{\blacktriangle} = X^{\blacktriangledown\blacktriangle} \cup \mathcal{B}^c = X^{\blacktriangledown} \cup \mathcal{B}^c \tag{3.3}$$

Lemma 3.2. *Let R be any reflexive relation on the set U and $(X, Y) \in \mathcal{R}_c^*$.*

(i) *If $\mathcal{B}^c \subseteq Y$, then $X \cup \mathcal{B}^c = Y$;*
(ii) *If $X \subseteq \mathcal{B}$, then $Y \cap \mathcal{B} = X$.*

Proof. (i) If $\mathcal{B}^c \subseteq Y$. Then, $Y \cup \mathcal{B}^c = Y$. Since $(X, Y) \in \mathcal{R}_c^*$, by Equation 3.1, we have

$$X \cap \mathcal{B} = Y \cap \mathcal{B} \Rightarrow (X \cap \mathcal{B}) \cup \mathcal{B}^c = (Y \cap \mathcal{B}) \cup \mathcal{B}^c \Rightarrow X \cup \mathcal{B}^c = Y.$$

(ii) If $X \subseteq \mathcal{B}$. Then, $X \cap \mathcal{B} = X$.

$$(X, Y) \in \mathcal{R}_c^* \Rightarrow X \cap \mathcal{B} = Y \cap \mathcal{B} \Rightarrow X = Y \cap \mathcal{B}.$$

□

In the case of reflexive relation, whether the ordered pair $(\mathcal{B}, U)$ is in $\mathcal{R}^*$ or not cannot be determined, as there is no characterization for $\mathcal{R}^*$. But $(\mathcal{B}, U) \in \mathcal{R}_c^*$ can be determined as $\mathcal{B}^{\triangle\blacktriangle} \subseteq U$. Because $(\mathcal{R}_c^*, \leq)$ is a symmetric lattice, $\sim (\mathcal{B}, U) = (\emptyset, \mathcal{B}^c) \in \mathcal{R}_c^*$. Now, $((\mathcal{B}, U)]$ is a principal ideal, and $[(\emptyset, \mathcal{B}^c))$ is a principal filter of $(\mathcal{R}_c^*, \leq)$.

Proposition 3.3. *Let R be any reflexive relation on the set U. Then*

$$\wp(U)^{\blacktriangle} \cong ((\mathcal{B}, U)] \text{ and } \wp(U)^{\blacktriangledown} \cong [(\emptyset, \mathcal{B}^c)) \text{ in } (\mathcal{R}_c^*, \leq).$$

Proof. Let $\varphi : \wp(U)^{\blacktriangle} \to ((\mathcal{B}, U)]$ be defined by

$$\varphi(X^{\blacktriangle}) = ((X^{\blacktriangle} \cap \mathcal{B})^{\blacktriangledown}, X^{\blacktriangle}) = (X^{\blacktriangle} \cap \mathcal{B}, X^{\blacktriangle}),$$

by Equation (3.2). First, we prove that $(X^{\blacktriangle} \cap \mathcal{B}, X^{\blacktriangle}) \in \mathcal{R}_c^*$. Let $x \in (X^{\blacktriangle} \cap \mathcal{B})^{\triangle\blacktriangle}$. Then $R(x) \cap (X^{\blacktriangle} \cap \mathcal{B})^{\triangle} \neq \emptyset$. This implies $\exists y \in U$ such that $y \in R(x)$ and $y \in (X^{\blacktriangle} \cap \mathcal{B})^{\triangle} \Rightarrow R^{-1}(y) \cap (X^{\blacktriangle} \cap \mathcal{B}) \neq \emptyset$ $\Rightarrow \exists z \in U$, such that $z \in R^{-1}(y)$ and $z \in X^{\blacktriangle} \cap \mathcal{B} \Rightarrow y \in R(z), z \in X^{\blacktriangle}$ and $z \in \mathcal{B} \Rightarrow \{z\} = R(z) \subseteq X$ and $z = y \Rightarrow \exists y \in U$ such that $y \in R(x)$ and $y \in X \Rightarrow R(x) \cap X \neq \emptyset \Rightarrow x \in X^{\blacktriangle}$. Therefore $(X^{\blacktriangle} \cap \mathcal{B})^{\triangle\blacktriangle} \subseteq X^{\blacktriangle}$. Hence

$$(X^{\blacktriangle} \cap \mathcal{B}, X^{\blacktriangle}) \in \mathcal{R}_c^* \text{ and } (X^{\blacktriangle} \cap \mathcal{B}, X^{\blacktriangle}) \leq (\mathcal{B}, U).$$

Next, we prove φ is an order isomorphism. Let $X^{\blacktriangle}, Y^{\blacktriangle} \in \wp(U)^{\blacktriangle}$ such that

$$\begin{aligned} X^{\blacktriangle} \subseteq Y^{\blacktriangle} &\Leftrightarrow X^{\blacktriangle} \cap \mathcal{B} \subseteq Y^{\blacktriangle} \cap \mathcal{B} \\ &\Leftrightarrow (X^{\blacktriangle} \cap \mathcal{B}, X^{\blacktriangle}) \leq (Y^{\blacktriangle} \cap \mathcal{B}, Y^{\blacktriangle}) \\ &\Leftrightarrow \varphi(X^{\blacktriangle}) \leq \varphi(Y^{\blacktriangle}). \end{aligned}$$

Let $(X, Y) \in ((\mathcal{B}, U)]$ in $\mathcal{R}_c^*$. This implies $X \subseteq \mathcal{B}$. Then, by Lemma 3.2(ii), we have $X = Y \cap \mathcal{B}$. Therefore for every $(X, Y) \in ((\mathcal{B}, U)]$, $\exists Y \in \wp(U)^{\blacktriangle}$ such that $\varphi(Y) = (Y \cap \mathcal{B}, Y) = (X, Y)$. Therefore φ is an order isomorphism and hence $\wp(U)^{\blacktriangle} \cong ((\mathcal{B}, U)]$. Dually, we have

$$(Y^{\blacktriangledown}, (Y^{\blacktriangledown} \cup \mathcal{B}^c)^{\blacktriangle}) = (Y^{\blacktriangledown}, Y^{\blacktriangledown} \cup \mathcal{B}^c) \in \mathcal{R}_c^*$$

and $(\emptyset, \mathcal{B}^c) \leq (Y^\blacktriangledown, Y^\blacktriangledown \cup \mathcal{B}^c)$. Therefore $\psi : \wp(U)^\blacktriangledown \to [(\emptyset, \mathcal{B}^c))$ defined by $\psi(Y^\blacktriangledown) = (Y^\blacktriangledown, (Y^\blacktriangledown \cup \mathcal{B}^c)^\blacktriangle)$ is order isomorphic and $\wp(U)^\blacktriangledown \cong [(\emptyset, \mathcal{B}^c))$. □

Lemma 3.4. $[(\emptyset, \mathcal{B}^c)) \cap ((\mathcal{B}, U)] = [(\emptyset, \mathcal{B}^c), (\mathcal{B}, U)] \cong 2^{\mathcal{B}}$ *in* $\mathcal{R}_c^*$.

Proof. For $X \subseteq \mathcal{B}$, $X^\vartriangle = X$. Let $x \in X^{\vartriangle\blacktriangle}$. This implies

$$x \in X^\blacktriangle \Rightarrow R(x) \cap X \neq \emptyset \Rightarrow \exists y \in U \text{ such that } y \in R(x) \text{ and } y \in X.$$

Now, we have two cases:

Case (i) If $y = x$, then $x = y \in X$ which implies $x \in X \cup \mathcal{B}^c$.

Case (ii) If $x \neq y$, then $x \in \mathcal{B}^c \Rightarrow x \in X \cup \mathcal{B}^c$.

Thus $X^{\vartriangle\blacktriangle} \subseteq X \cup \mathcal{B}^c$ and $(X \cup \mathcal{B}^c) \cap \mathcal{B} = X \cap \mathcal{B}$. So, for any $X \subseteq \mathcal{B}$, $(X, X \cup \mathcal{B}^c) \in [(\emptyset, \mathcal{B}^c), (\mathcal{B}, U)]$ in $\mathcal{R}_c^*$. Let us define a map

$$g : 2^{\mathcal{B}} \to [(\emptyset, \mathcal{B}^c)) \cap ((\mathcal{B}, U)]$$

by $g(X) = (X, X \cup \mathcal{B}^c)$. Let $X, Y \subseteq \mathcal{B}$ such that

$$X \subseteq Y \Leftrightarrow (X, X \cup \mathcal{B}^c) \leq (Y, Y \cup \mathcal{B}^c) \Leftrightarrow g(X) \leq g(Y).$$

Let $(X, Y) \in [(\emptyset, \mathcal{B}^c), (\mathcal{B}, U)]$ in $\mathcal{R}_c^*$. Then $\mathcal{B}^c \subseteq Y$, which implies $Y = X \cup \mathcal{B}^c$ by Lemma 3.2(i). Now, $g(X) = (X, X \cup \mathcal{B}^c) = (X, Y)$. Therefore g is an order isomorphism. Hence $[(\emptyset, \mathcal{B}^c), (\mathcal{B}, U)] \cong 2^{\mathcal{B}}$. □

The following illustration explains the Proposition 3.3. Let us denote the sets like $\{a, b\}$ by ab, except for U in all the following Examples.

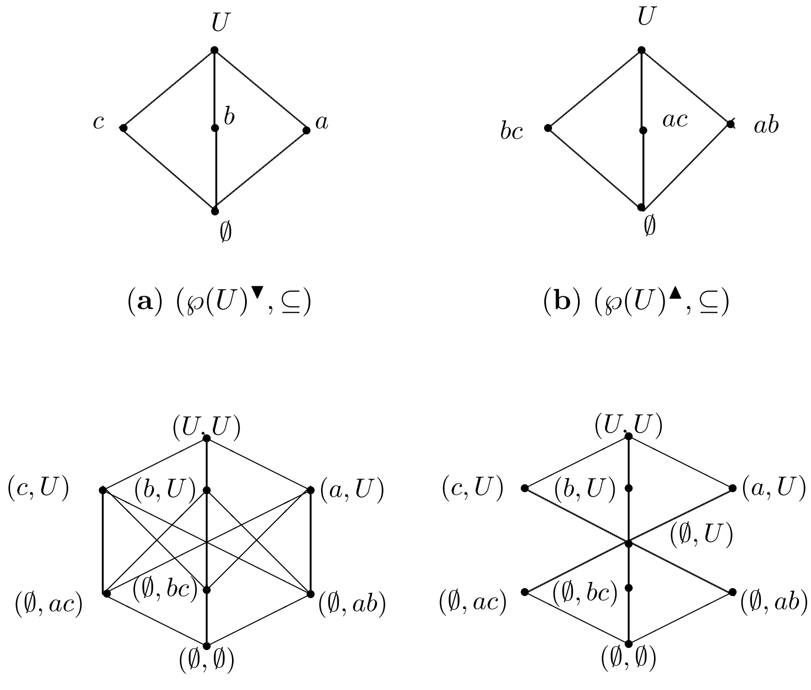
Example 3.5. Let us consider a reflexive relation

$$R = \{aRa, , bRb, cRc, aRb, bRc, cRa\}$$

on a set $U = \{a, b, c\}$. Then

$$\mathcal{R}^* = \{(\emptyset, \emptyset), (\emptyset, ab), (\emptyset, bc), (\emptyset, ac), (a, U), (b, U), (c, U), (U, U)\}$$

and $(\mathcal{R}^*, \leq)$ is not a lattice as shown in [9]. Here $\mathcal{B} = \emptyset$. $\mathcal{R}_c^*$ is given by $\mathcal{R}^* \cup \{(\emptyset, U)\}$. One can see in Figure 1 (d) below that $\wp(U)^\blacktriangle \cong ((\emptyset, U)]$ and $\wp(U)^\blacktriangledown \cong [(\emptyset, U))$ in $(\mathcal{R}_c^*, \leq)$.



(**a**) $(\wp(U)^{\blacktriangledown}, \subseteq)$ (**b**) $(\wp(U)^{\blacktriangle}, \subseteq)$

(**c**) $(\mathcal{R}^*, \leq)$ (**d**) $(\mathcal{R}_c^*, \leq)$

Figure 1.

Let P be an ordered set and $S \subseteq P$. Then S is called Join-dense in P if, for every element $a \in P$, there exists a subset A of S such that $a = \vee A$. The dual of Join-dense is Meet-dense [5].

$$\mathcal{J}_{Ref} = \{(\emptyset, \{x\}^{\blacktriangle}) | x \in U \text{ and } |R(x)| \geq 2\} \bigcup \{(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) | x \in U\} \quad (3.4)$$

Proposition 3.6. *$\mathcal{J}_{Ref}$ is Join-dense in $(\mathcal{R}^*, \leq)$.*

Proof. Let $x \in U$. If $|R(x)| \geq 2$, then $\{x\}^{\blacktriangledown} = \emptyset$. This implies $(\{x\}^{\blacktriangledown}, \{x\}^{\blacktriangle}) = (\emptyset, \{x\}^{\blacktriangle}) \in \mathcal{R}^*$ and for all $x \in U$, $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \in \mathcal{R}^*$. Thus $\mathcal{J}_{Ref} \subseteq \mathcal{R}^*$. Let $(X^{\blacktriangledown}, X^{\blacktriangle}) \in \mathcal{R}^*$ and

$$\mathcal{A} = \{(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) | \{x\}^{\vartriangle} \subseteq X\} \cup \{(\emptyset, \{x\}^{\blacktriangle}) | x \in X \ \& \ |R(x)| \geq 2\}.$$

Clearly, $\mathcal{A} \subseteq \mathcal{R}^*$. We have from Lemma 3.1, $\bigcup\{\{x\}^{\vartriangle\blacktriangledown} | \{x\}^{\vartriangle} \subseteq X\} = X^{\blacktriangledown}$. Also, we have $\bigcup\{\{x\}^{\vartriangle} | \{x\}^{\vartriangle} \subseteq X\} \subseteq X$ and the upper approximation operator distributes over unions. This implies $\bigcup\{\{x\}^{\vartriangle\blacktriangle} | \{x\}^{\vartriangle} \subseteq X\} \subseteq X^{\blacktriangle}$. Therefore we have

$$\bigvee\{(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) | \{x\}^{\vartriangle} \subseteq X\} \leq (X^{\blacktriangledown}, X^{\blacktriangle}).$$

Obviously, $\bigvee\{(\emptyset, \{x\}^{\blacktriangle}) | x \in X \ \& \ |R(x)| \geq 2\} \leq (X^{\blacktriangledown}, X^{\blacktriangle})$. Hence $\bigvee \mathcal{A} \leq (X^{\blacktriangledown}, X^{\blacktriangle})$. We claim that, $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq \bigvee \mathcal{A}$. Let $x \in X^{\blacktriangle}$. Then there are two cases:

Case (i) If $x \in B$, then $x \in X^{\blacktriangledown}$. This implies $\{x\}^{\triangle} = R(x) \subseteq X$. We have $R(x) \subseteq \{x\}^{\triangle}$ which implies $x \in \{x\}^{\triangle\blacktriangledown}$ and $x \in \{x\}^{\triangle\blacktriangle}, \{x\}^{\triangle} \subseteq X$. Thus $(\{x\}^{\triangle\blacktriangledown}, \{x\}^{\triangle\blacktriangle}) \in \mathcal{A}$.

Case (ii) If $x \in B^c$, then $|R(x)| \geq 2$ and so $R(x) = \{x\}^{\triangle}$ contains atleast two elements including x. This implies $\{x\}^{\blacktriangledown} = \emptyset$ and $x \in \{x\}^{\blacktriangle}$. Thus $(\{x\}^{\blacktriangledown}, \{x\}^{\blacktriangle}) = (\emptyset, \{x\}^{\blacktriangle}) \in \mathcal{A}$. Therefore, we have $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq \bigvee \mathcal{A}$. So for every $(X^{\blacktriangledown}, X^{\blacktriangle}) \in \mathcal{R}^*$, $\exists \mathcal{A} \subseteq \mathcal{J}_{Ref}$ such that $(X^{\blacktriangledown}, X^{\blacktriangle}) = \bigvee \mathcal{A}$. Hence $\mathcal{J}_{Ref}$ is Join-dense in $(\mathcal{R}^*, \leq)$. □

Example 3.7. Consider a reflexive relation

$$R = \{aRa, aRb, aRc, bRa, bRb, cRa, cRc, dRd\}$$

on the set $U = \{a, b, c, d\}$. Then $R(a) = \{a, b, c\}$, $R(b) = \{a, b\}$, $R(c) = \{a, c\}$, $R(d) = \{d\}$. The elements of the sets $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are listed in Table 1.

Table 1. The Elements of $\wp(U)^{\blacktriangledown}$ & $\wp(U)^{\blacktriangle}$

$\wp(U)$	$\wp(U)^{\blacktriangledown}$	$\wp(U)^{\blacktriangle}$
$\emptyset$	$\emptyset$	$\emptyset$
a	$\emptyset$	abc
b	$\emptyset$	ab
c	$\emptyset$	ac
d	d	d
ab	b	abc
bc	$\emptyset$	abc
cd	d	acd
ad	d	U
bd	d	abd
ac	c	abc
abc	abc	abc
bcd	d	U
acd	cd	U
abd	bd	U
U	U	U

Therefore

$$\mathcal{R}^* = \mathcal{R}^*_c = \{(\emptyset, \emptyset), (U, U), (\emptyset, ac), (\emptyset, ab), (\emptyset, abc), (d, d), (d, acd), (d, abd), (d, U), (b, abc), (c, abc), (cd, U), (abc, abc), (bd, U)\}.$$

Here $\mathcal{B} = \{d\}$. The Hasse diagram of all the above-mentioned sets with their partial order is shown in Figure 2. It can be noted that $\wp(U)^{\blacktriangle} \cong ((d, U)]$ and $\wp(U)^{\blacktriangledown} \cong [(\emptyset, abc))$ in $(\mathcal{R}_c^*, \leq)$.

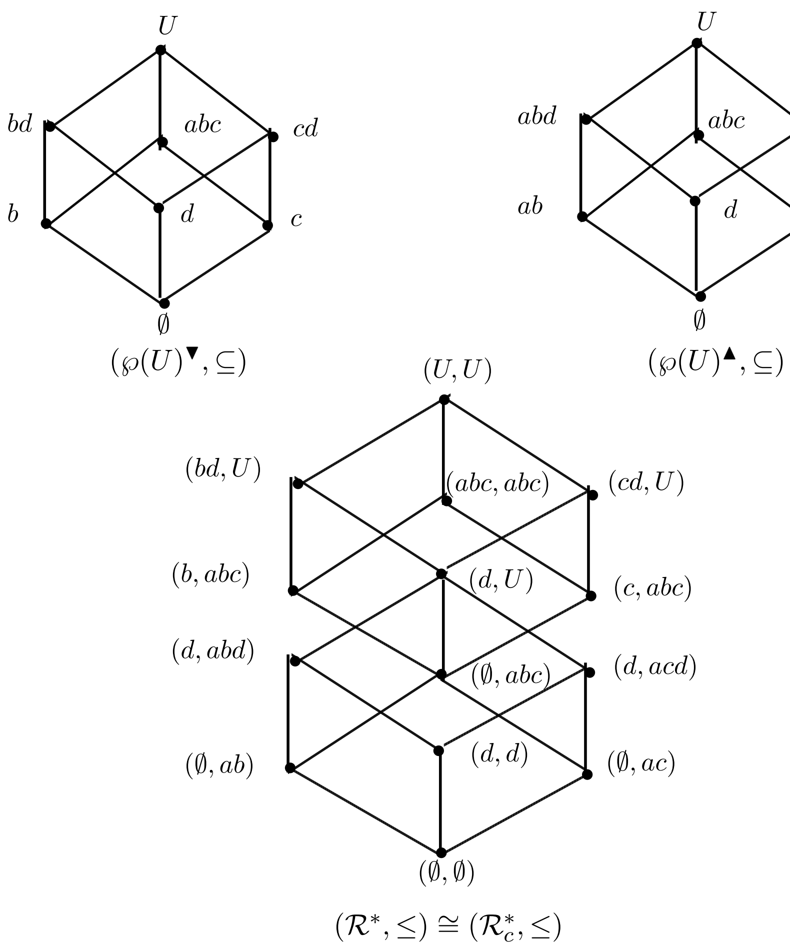


Figure 2.

Remark 3.8. For a reflexive relation R on U, $\mathcal{J}_{Ref}$ is Join-dense in $(\mathcal{R}^*, \leq)$. Because $(\mathcal{R}_c^*, \leq)$ is a completion of $(\mathcal{R}^*, \leq)$, every non-zero element of $\mathcal{R}_c^*$ can be written as a join of elements from $\mathcal{J}_{Ref}$. If R is a quasi-order relation on U, then $\mathcal{J}_{Ref}$ is the set of all join-irreducible elements of $\mathcal{R}^*$. This was proved by Järvinen in [11]. But for merely a reflexive relation, the elements of $\mathcal{J}_{Ref}$ are not necessarily join-irreducible. It is evident from the above Example 3.7 that $\mathcal{J}_{Ref} = \{(\emptyset, abc), (\emptyset, ab), (\emptyset, ac), (d, d), (abc, abc), (b, abc), (c, abc)\}$. Here the element $(\emptyset, abc) \in \mathcal{J}_{Ref}$ can be obtained as the join of $(\emptyset, ab)$ and $(\emptyset, ac)$. Similarly, the element $(abc, abc) \in \mathcal{J}_{Ref}$ also be obtained as the join of (b, abc) and (c, abc). Hence both elements are not join-irreducible.

4. Pasting of Rough Approximation lattices

The pasting of rough approximation lattices together over a common Boolean sublattice and its relation to the lattice structure of rough sets system defined by a quasi-order relation was studied in [17]. Here, the pasting of lattices $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$ together over a common Boolean sublattice is studied in the generalized case of reflexive relation.

Dilworth Gluing:[1] If a non-empty filter F_1 of a lattice P_1 is isomorphic to an ideal F_2 of a lattice P_2, let P be the union of P_1 and P_2 with the elements of F_1 and F_2 identified via the isomorphism. P can be ordered with the transitive closure of the union of the orders on P_1 and P_2. It is easy to see that under this order P is a lattice.

A set $S \subseteq P$ is said to be upward-closed in P, for every $x \in P$, if $s \leq x \Rightarrow x \in S, \forall s \in S$. Similarly, a set $S \subseteq P$ is said to be downward-closed in P, for every $x \in P$, if $x \leq s \Rightarrow x \in S$, $\forall s \in S$.

Result 4.1. [1] Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be complete lattices. Let $(P, \leq)$ be the gluing of P_1 and P_2. If $D = P_1 \cap P_2 \neq \emptyset$ is upward-closed in $(P_1, \leq_1)$ and is downward-closed in $(P_2, \leq_2)$. Then D and $(P, \leq)$ are complete lattices.

Let R be a reflexive relation on a set U. If $X \subseteq \mathcal{B}$. Let

$$x \in X \Rightarrow \{x\} = R(x) \subseteq X \Rightarrow x \in X^{\blacktriangledown}.$$

Also, we have $X^{\blacktriangledown} \subseteq X$. Therefore $X = X^{\blacktriangledown}$.

$$X \in \wp(\mathcal{B}) \Leftrightarrow X \subseteq \mathcal{B} \Leftrightarrow X = X^{\blacktriangledown} \subseteq \mathcal{B} \Leftrightarrow X \in (\mathcal{B}] \text{ in } (\wp(U)^{\blacktriangledown}, \subseteq)$$

Therefore the principal ideal $(\mathcal{B}] = \wp(\mathcal{B})$ in $(\wp(U)^{\blacktriangledown}, \subseteq)$ is a Boolean sublattice with the least element $\emptyset$ and the greatest element $\mathcal{B}$. Dually, the principal filter $[\mathcal{B}^c)$ in $(\wp(U)^{\blacktriangle}, \subseteq)$ is also a Boolean sublattice with the least element $\mathcal{B}^c$ and the greatest element U. Thus $(\mathcal{B}] \cong [\mathcal{B}^c) \cong 2^{\mathcal{B}}$.

Let the principal ideal $(\mathcal{B}]$ of $(\wp(U)^{\blacktriangledown}, \subseteq)$ be glued over the principal filter $[\mathcal{B}^c)$ of $(\wp(U)^{\blacktriangle}, \subseteq)$ [via the mappings $(X^{\blacktriangledown})^* \rightarrow X^{\blacktriangledown\blacktriangle} \cup \mathcal{B}^c$ and $(X^{\blacktriangle})_* \rightarrow X^{\blacktriangle\blacktriangledown} \cap \mathcal{B}$]. The identified elements form a Boolean lattice isomorphic to $2^{\mathcal{B}}$. Let this Boolean part be denoted by $\mathbf{S} = (\mathbf{S}, \sqcup_S, \sqcap_S)$ throughout the paper, where $\sqcup_S$ and $\sqcap_S$ are the join and meet operations in $\mathbf{S}$ respectively.

Let $S \in \mathbf{S}$. For every $X^{\blacktriangle} \in \wp(U)^{\blacktriangle}$, if $S \subseteq X^{\blacktriangle}$, then $S \in \mathbf{S} \Rightarrow \mathcal{B}^c \subseteq S \subseteq X^{\blacktriangle}$. Thus $X^{\blacktriangle} \in [\mathcal{B}^c)$ implies $X^{\blacktriangle} \in \mathbf{S}$. Therefore $\mathbf{S}$ is upward-closed in $(\wp(U)^{\blacktriangle}, \subseteq)$. For every $X^{\blacktriangledown} \in \wp(U)^{\blacktriangledown}$, if $X^{\blacktriangledown} \subseteq S$, then $S \in \mathbf{S} \Rightarrow S \subseteq \mathcal{B}$ which implies $X^{\blacktriangledown} \subseteq \mathcal{B} \Rightarrow X^{\blacktriangledown} \in \mathbf{S}$. Therefore $\mathbf{S}$ is downward-closed in $(\wp(U)^{\blacktriangledown}, \subseteq)$. Also, $\mathbf{S}$ is always non-empty. Because, in the case of $\mathcal{B} = \emptyset$ also, U of $(\wp(U)^{\blacktriangle}, \subseteq)$

is glued over $\emptyset$ of $(\wp(U)^{\blacktriangledown}, \subseteq)$. Hence by Result 4.1, we have the following proposition.

Proposition 4.2. *For a reflexive relation R on U, the gluing of $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$ over $\boldsymbol{S}$ is a complete lattice.*

Definition 4.3. Let L be a lattice. Let A, B, S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Let f_A and f_B be the embeddings of A and B respectively into L. Then L pastes A and B together over S in notation $L = Paste(A, B, S)$, if whenever g_A and g_B are embeddings of A and B into a lattice K satisfying $xg_A = xg_B$ for all $x \in S$, then there is a homomorphism h of L into K satisfying $f_A h = g_A$ and $f_B h = g_B$.

Lemma 4.4. [4] *Let C, D, and S be lattices, $C \cap D = S$. On $P = C \cup D$, we define a binary relation $\leq$ as follows:*

(i) *for $x, y \in C$(and for $x, y \in D$), $x \leq y$ in P if and only if $x \leq y$ in C (respectively, $x \leq y$ in D);*
(ii) *for $x \in C$ and for $y \in D$, $x \leq y$ in P iff $\exists s \in S$ with $x \leq s$ in C and $s \leq y$ in D ; and dually, for $y \leq x$.*

Then the pasting of C and D together over S is a poset and is denoted by $P(C, D, S)$.

The poset $P(C, D, S)$ may be a lattice. If L pastes C and D together over S, then L as a poset is isomorphic to $P(C, D, S)$, but not the converse.

In a poset $(P, \leq)$, an element b is said to *cover* an element a $(a \prec b)$ if $a < b$ and there exists no c with $a < c < b$.

Theorem 4.5. [6] *Let L be a finite lattice. Let C, D, S be sublattices of L, $C \cap D = S$, $C \cup D = L$. L pastes C and D together over S iff the following two conditions hold:*

(1) *For $a \in C$ and $b \in D$, if $a < b$, then there exists an $s \in S$ satisfying $a \leq s \leq b$; and dually.*
(2) *For $s \in S$, all the covers of s in L are in C or all are in D; and dually.*

Every gluing is a pasting. Considering pasting as a generalized one, here onwards the pasting of $\wp(U)^{\blacktriangle}$ and $\wp(U)^{\blacktriangledown}$ together over $\mathbf{S}$ is discussed and is denoted by $\mathbb{P} = P(\wp(U)^{\blacktriangle}, \wp(U)^{\blacktriangledown}, \mathbf{S})$. Let the ordering in $\mathbb{P}$ be denoted by $\leq_P$ and is defined as follows

(i) For $X^{\blacktriangle}, Y^{\blacktriangle} \in \wp(U)^{\blacktriangle}$, $X^{\blacktriangle} \leq_P Y^{\blacktriangle} \Rightarrow X^{\blacktriangle} \subseteq Y^{\blacktriangle}$.
(ii) For $X^{\blacktriangledown}, Y^{\blacktriangledown} \in \wp(U)^{\blacktriangledown}$, $X^{\blacktriangledown} \leq_P Y^{\blacktriangledown} \Rightarrow X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$.

(iii) For $X^\blacktriangle \in \wp(U)^\blacktriangle$ and $Y^\blacktriangledown \in \wp(U)^\blacktriangledown$, $X^\blacktriangle \leq_P Y^\blacktriangledown \Rightarrow \exists S \in \mathbf{S}$, such that $X^\blacktriangle \subseteq S^*$ in $\wp(U)^\blacktriangle$ and $S_* \subseteq Y^\blacktriangledown$ in $\wp(U)^\blacktriangledown$.

Example 4.6. Consider a reflexive relation

$$R = \{aRa, aRb, aRc, aRd, bRb, bRc, cRc, cRd, dRd\}$$

on $U = \{a, b, c, d\}$. Then

$$\wp(U)^\blacktriangledown = \{\emptyset, b, d, cd, bcd, U\},\ \wp(U)^\blacktriangle = \{\emptyset, a, ab, abc, acd, U\}$$

and

$$\mathcal{R}_c^* = \mathcal{R}^* = \{(\emptyset,\emptyset), (U,U), (\emptyset, a), (\emptyset, ab), (\emptyset, abc), (d, acd), (d, U), (b, abc), (cd, U), (bcd, U)\}.$$

Here $\mathcal{B} = \{d\}$. The pasting of $\wp(U)^\blacktriangledown$ and $\wp(U)^\blacktriangle$ over $\mathbf{S}$ shown in Figure 3, is the same as $(\mathcal{R}^*, \leq)$ and $(\mathcal{R}_c^*, \leq)$.

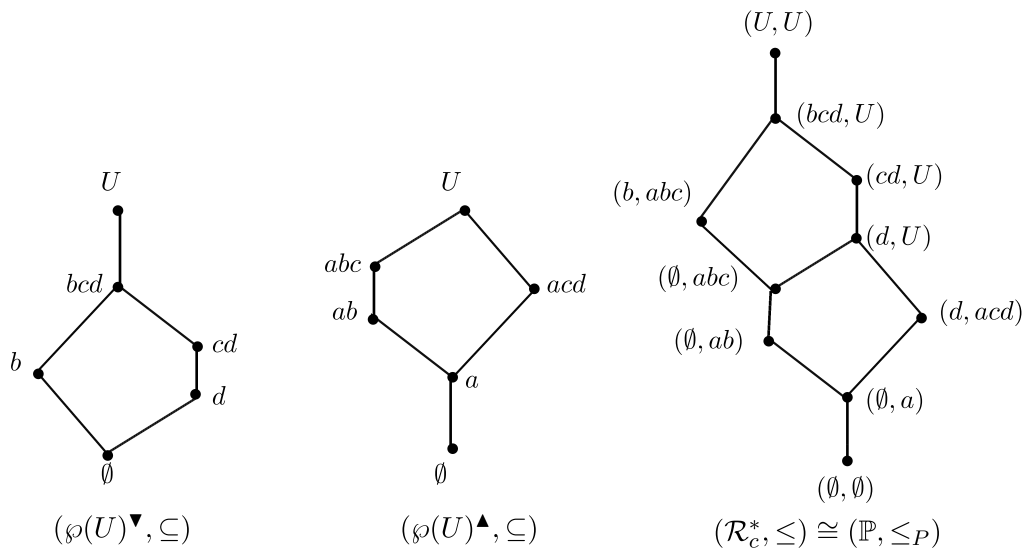


Figure 3.

Proposition 4.7. *For a reflexive relation R on U, $(\mathbb{P}, \leq_P)$ is order embedded in $(\mathcal{R}_c^*, \leq)$.*

Proof. $\mathbb{P} = \wp(U)^\blacktriangle \cup \wp(U)^\blacktriangledown$ and $\mathbf{S} = (\mathcal{B}]$ $(\cong [\mathcal{B}^c))$. Let $h : \mathbb{P} \to \mathcal{R}_c^*$ be defined by

$$h(A) = \begin{cases} ((X^\blacktriangle \cap \mathcal{B})^\blacktriangledown, X^\blacktriangle) & \text{if } A = X^\blacktriangle \\ (X^\blacktriangledown, (X^\blacktriangledown \cup \mathcal{B}^c)^\blacktriangle) & \text{if } A = X^\blacktriangledown \end{cases}$$

$((X^\blacktriangle \cap \mathcal{B})^\blacktriangledown, X^\blacktriangle), (X^\blacktriangledown, (X^\blacktriangledown \cup \mathcal{B}^c)^\blacktriangle) \in \mathcal{R}_c^*$ is already proved in Proposition 3.3. Let $A, C \in \mathbb{P}$ such that $A \leq_P C$. If both A and C are in $(\wp(U)^\blacktriangledown, \subseteq)$ or in $(\wp(U)^\blacktriangle, \subseteq)$, then trivially $A \leq_P C \Rightarrow h(A) \leq h(C)$.

If $A = X^\blacktriangle$ and $C = Y^\blacktriangledown$, then $A \leq_P C \Leftrightarrow X^\blacktriangle \leq_P Y^\blacktriangledown$. Then $\exists S \in \mathbf{S}$ such that $X^\blacktriangle \subseteq S^*$ in $(\wp(U)^\blacktriangle, \subseteq)$ and $S_* \subseteq Y^\blacktriangledown$ in $(\wp(U)^\blacktriangledown, \subseteq)$.

$$\begin{aligned} X^\blacktriangle \subseteq S^* \text{ in } (\wp(U)^\blacktriangle, \subseteq) &\Rightarrow X^\blacktriangle \subseteq S \cup \mathcal{B}^c \\ &\Rightarrow X^\blacktriangle \cap \mathcal{B} \subseteq (S \cup \mathcal{B}^c) \cap \mathcal{B} = S \cap \mathcal{B} = S_* \subseteq Y^\blacktriangledown. \end{aligned}$$

By Equation (3.2), $(X^\blacktriangle \cap \mathcal{B})^\blacktriangledown = X^\blacktriangle \cap \mathcal{B} \subseteq Y^\blacktriangledown$. Similarly from $S_* \subseteq Y^\blacktriangledown$ in $(\wp(U)^\blacktriangledown, \subseteq)$, we have $X^\blacktriangle \subseteq Y^\blacktriangledown \cup \mathcal{B}^c = (Y^\blacktriangledown \cup \mathcal{B}^c)^\blacktriangle$ by Equation (3.3). Therefore $((X^\blacktriangle \cap \mathcal{B})^\blacktriangledown, X^\blacktriangle) \leq (Y^\blacktriangledown, (Y^\blacktriangledown \cup \mathcal{B}^c)^\blacktriangle) \Rightarrow h(A) \leq h(C)$.

Suppose

$$\begin{aligned} h(A) \leq h(C) &\Rightarrow ((X^\blacktriangle \cap \mathcal{B})^\blacktriangledown, X^\blacktriangle) \leq (Y^\blacktriangledown, (Y^\blacktriangledown \cup \mathcal{B}^c)^\blacktriangle) \\ &\Rightarrow X^\blacktriangle \cap \mathcal{B} \subseteq Y^\blacktriangledown \text{ and } X^\blacktriangle \subseteq Y^\blacktriangledown \cup \mathcal{B}^c. \end{aligned}$$

Let $S = (X^\blacktriangle \cup \mathcal{B}^c) \sqcap_S (Y^\blacktriangledown \cap \mathcal{B})$. Clearly $S \in \mathbf{S}$,

$$S^* = (X^\blacktriangle \cup \mathcal{B}^c) \cap (Y^\blacktriangledown \cap \mathcal{B})^* \text{ and } S_* = (X^\blacktriangle \cup \mathcal{B}^c)_* \cap (Y^\blacktriangledown \cap \mathcal{B}).$$

By assumption, $X^\blacktriangle \subseteq Y^\blacktriangledown \cup \mathcal{B}^c = (Y^\blacktriangledown \cap \mathcal{B})^*$. We have $X^\blacktriangle \subseteq X^\blacktriangle \cup \mathcal{B}^c$ and $X^\blacktriangle \subseteq (Y^\blacktriangledown \cap \mathcal{B})^*$ in $\wp(U)^\blacktriangle$. That implies $X^\blacktriangle \subseteq (X^\blacktriangle \cup \mathcal{B}^c) \cap (Y^\blacktriangledown \cap \mathcal{B})^* = S^*$ in $\wp(U)^\blacktriangle$. Similarly, $(X^\blacktriangle \cup \mathcal{B}^c)_* = X^\blacktriangle \cap \mathcal{B} \subseteq Y^\blacktriangledown$, by assumption. We have, $Y^\blacktriangledown \cap \mathcal{B} \subseteq Y^\blacktriangledown$ and $(X^\blacktriangle \cup \mathcal{B}^c)_* \subseteq Y^\blacktriangledown$ in $\wp(U)^\blacktriangledown$, which implies $S_* = (X^\blacktriangle \cup \mathcal{B}^c)_* \cap (Y^\blacktriangledown \cap \mathcal{B}) \subseteq Y^\blacktriangledown$ in $\wp(U)^\blacktriangledown$. Hence $X^\blacktriangle \leq_P Y^\blacktriangledown$ in $\mathbb{P}$. Therefore h is an order embedding and hence $(\mathbb{P}, \leq_P)$ is order embedded in $(\mathcal{R}_c^*, \leq)$. □

Example 4.8. Consider a reflexive relation

$$R = \{aRa, aRc, bRb, bRd, cRa, cRc, cRd, dRb, dRc, dRd\}$$

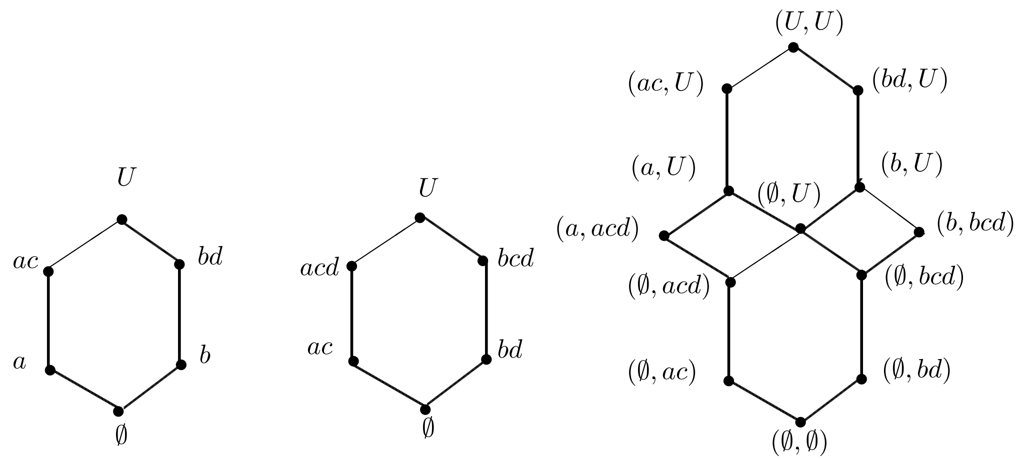
on $U = \{a, b, c, d\}$. Then

$$\wp(U)^\blacktriangledown = \{\emptyset, a, b, ac, bd, U\},\ \wp(U)^\blacktriangle = \{\emptyset, ac, bd, bcd, acd, U\}$$

and

$$\begin{aligned} \mathcal{R}^* = \mathcal{R}_c^* = \{&(\emptyset,\emptyset), (U,U), (\emptyset, ac), (\emptyset, bd), (\emptyset, acd), (\emptyset, bcd), (\emptyset, U), \\ &(a, acd), (b, bcd), (a, U), (b, U), (ac, U), (bd, U)\}. \end{aligned}$$

Here $\mathcal{B} = \emptyset$. So, $\mathbf{S} = \{(\emptyset, U)\}$.



$(\wp(U)^{\blacktriangledown}, \subseteq)$ $(\wp(U)^{\blacktriangle}, \subseteq)$ $(\mathcal{R}_c^*, \leq) \cong (\mathcal{R}^*, \leq)$

Figure 4.

It can be seen trivially from Figure 4 that the pasting of $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$ over $\mathbf{S}$ is order embedded in $(\mathcal{R}_c^*, \leq) \cong (\mathcal{R}^*, \leq)$. But in Example 3.5, the pasting of $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$ over $\mathbf{S}$ is order embedded in $(\mathcal{R}_c^*, \leq)$, but not in $(\mathcal{R}^*, \leq)$.

Proposition 4.9. *Let R be any reflexive relation on the set U and $x \in \mathcal{B}^c$. Then the following are equivalent*

(i) $R(x) \cap R(y) \neq \emptyset, \forall y \in \mathcal{B}^c$
(ii) $\mathcal{B}^c \subseteq \{x\}^{\vartriangle\blacktriangle}$
(iii) $\mathcal{R}_c^* = [(\emptyset, \mathcal{B}^c)) \cup ((\mathcal{B}, U)]$

Proof. Let $x \in \mathcal{B}^c$.

(i)$\Rightarrow$ (ii) Assume $R(x) \cap R(y) \neq \emptyset, \forall y \in \mathcal{B}^c$ holds. To show, $\mathcal{B}^c \subseteq \{x\}^{\vartriangle\blacktriangle}$. Let $y \in \mathcal{B}^c$. By assumption, we have

$$R(x) \cap R(y) \neq \emptyset \Rightarrow R(y) \cap \{x\}^{\vartriangle} \neq \emptyset \Rightarrow y \in \{x\}^{\vartriangle\blacktriangle}$$

Therefore $\mathcal{B}^c \subseteq \{x\}^{\vartriangle\blacktriangle}$.

(ii)$\Rightarrow$ (iii) Assume $\mathcal{B}^c \subseteq \{x\}^{\vartriangle\blacktriangle}$. Now it is to be proved,

$$\mathcal{R}_c^* = [(\emptyset, \mathcal{B}^c)) \cup ((\mathcal{B}, U)].$$

Since every element in $\mathcal{R}_c^*$ can be written as the join of elements from $\mathcal{J}_{Ref}$ and $[(\emptyset, \mathcal{B}^c))$, $((\mathcal{B}, U)]$ are sublattices of $\mathcal{R}_c^*$. It is enough to show that $\mathcal{J}_{Ref} \subseteq [(\emptyset, \mathcal{B}^c)) \cup ((\mathcal{B}, U)]$. Always $(\emptyset, \{x\}^{\blacktriangle}) \in ((\mathcal{B}, U)]$, for every $x \in U$ such that $|R(x)| \geq 2$. Let $x \in U$.

Case (i) If $x \in \mathcal{B}$, then $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) = (\{x\}, \{x\}^{\blacktriangle}) \in ((\mathcal{B}, U)]$.

Case (ii) If $x \in \mathcal{B}^c$, then by assumption, $\mathcal{B}^c \subseteq \{x\}^{\vartriangle\blacktriangle}$. This implies $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \in [(\emptyset, \mathcal{B}^c))$. Therefore $\mathcal{J}_{Ref} \subseteq [(\emptyset, \mathcal{B}^c)) \cup ((\mathcal{B}, U)]$.

(iii)$\Rightarrow$ (i) Next, assume $\mathcal{R}_c^* = [(\emptyset, \mathcal{B}^c)) \cup ((\mathcal{B}, U)]$ and let $x \in \mathcal{B}^c$. To show that, $R(x) \cap R(y) \neq \emptyset, \forall y \in \mathcal{B}^c$. We have $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \in \mathcal{R}_c^*$. If $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \in ((\mathcal{B}, U)]$, then $x \in \{x\}^{\vartriangle\blacktriangledown} \subseteq \mathcal{B} \Rightarrow x \in \mathcal{B}$, which is a contradiction to $x \in \mathcal{B}^c$. So by assumption, $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \in [(\emptyset, \mathcal{B}^c))$

$$\begin{aligned} &\Rightarrow \mathcal{B}^c \subseteq \{x\}^{\vartriangle\blacktriangle} \\ &\Rightarrow y \in \{x\}^{\vartriangle\blacktriangle}, \forall y \in \mathcal{B}^c \\ &\Rightarrow R(y) \cap \{x\}^{\vartriangle} \neq \emptyset, \forall y \in \mathcal{B}^c \\ &\Rightarrow R(y) \cap R(x) \neq \emptyset, \forall y \in \mathcal{B}^c. \end{aligned}$$

Hence Proved. □

Let the condition on R in the above proposition be defined as follows.

Definition 4.10. A reflexive relation R on U is said to be $*$-connected if for every $x, y \in \mathcal{B}^c$, $R(x) \cap R(y) \neq \emptyset$.

Remark 4.11. Suppose R is not $*$-connected, then for some $x \in \mathcal{B}^c$, $\exists y \in \mathcal{B}^c$ such that $R(x) \cap R(y) = \emptyset$ and $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \in \mathcal{R}_c^*$. But

$$(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \notin [(\emptyset, \mathcal{B}^c)) \cup ((\mathcal{B}, U)]$$

by Proposition 4.9. In the same way, it can be said that

$$(\{y\}^{\vartriangle\blacktriangledown}, \{y\}^{\vartriangle\blacktriangle}) \notin [(\emptyset, \mathcal{B}^c)) \cup ((\mathcal{B}, U)]$$

and $(\{y\}^{\vartriangle\blacktriangledown}, \{y\}^{\vartriangle\blacktriangle}) \in \mathcal{R}_c^*$. For instance, in Example 4.8 one can notice that the elements (a, acd) and (b, bcd) of $\mathcal{R}_c^*$ do not belong to both $((\mathcal{B}, U)]$ and $[(\emptyset, \mathcal{B}^c))$ in $\mathcal{R}_c^*$.

Theorem 4.12. *If R is $*$-connected, then $(\mathbb{P}, \leq_P)$ is order isomorphic to $(\mathcal{R}_c^*, \leq)$.*

Proof. It is shown in Proposition 4.7, that $(\mathbb{P}, \leq_P)$ is order embedded in $(\mathcal{R}_c^*, \leq)$. Therefore it is enough to show that the mapping h defined in Proposition 4.7 is onto. Let $(X, Y) \in \mathcal{R}_c^*$. Then (X, Y) will be in $[(\emptyset, \mathcal{B}^c))$ or in $((\mathcal{B}, U)]$, by Proposition 4.9. Suppose $(X, Y) \in [(\emptyset, \mathcal{B}^c))$. This implies $\mathcal{B}^c \subseteq Y \Rightarrow X \cup \mathcal{B}^c = Y$, by Lemma 3.2(i). Then for $X \in \wp(U)^{\blacktriangledown}$ in $\mathbb{P}$, $h(X) = (X, X \cup \mathcal{B}^c) = (X, Y)$. Suppose $(X, Y) \in ((\mathcal{B}, U)]$. This implies $X \subseteq \mathcal{B} \Rightarrow X = Y \cap \mathcal{B}$, by Lemma 3.2(ii). So $\exists Y \in \wp(U)^{\blacktriangle}$ in $\mathbb{P}$ such that $h(Y) = (Y \cap \mathcal{B}, Y) = (X, Y)$. Therefore h is onto and so h is an order isomorphism. Hence $(\mathbb{P}, \leq_P)$ is order isomorphic to $(\mathcal{R}_c^*, \leq)$. □

Theorem 4.13. *Let R be any reflexive relation on a finite set U. $\mathcal{R}_c^*$ pastes $((\mathcal{B}, U)]$ and $[(\emptyset, \mathcal{B}^c))$ together over $[(\emptyset, \mathcal{B}^c), (\mathcal{B}, U)]$ iff, R is $*$-connected.*

Proof. Assume R is $*$-connected. Then by Proposition 4.9, we have $\mathcal{R}_c^* = ((\mathcal{B}, U]] \cup [(\emptyset, \mathcal{B}^c))$ and let $\mathbf{S}' = [(\emptyset, \mathcal{B}^c), (\mathcal{B}, U)]$. Let $(A, B) \in ((\mathcal{B}, U)]$ and $(C, D) \in [(\emptyset, \mathcal{B}^c))$ such that $(A, B) \leq (C, D)$. Then $A \subseteq \mathcal{B}$, $\mathcal{B}^c \subseteq D$ and $A \subseteq C$, $B \subseteq D$. Let

$$I = (A, B \cup \mathcal{B}^c) \bigwedge (C \cap \mathcal{B}, D) = ((A \cap C) \cap \mathcal{B}, (B \cup \mathcal{B}^c) \cap D) = (A, B \cup \mathcal{B}^c)$$

(where $\bigwedge$ is the meet operation in $\mathcal{R}_c^*$). Then obviously, $I \in \mathbf{S}', (A, B) \leq I$ in $((\mathcal{B}, U)]$ and $I \leq (C, D)$ in $[(\emptyset, \mathcal{B}^c))$. Therefore for $(A, B) \leq (C, D)$, $\exists I \in \mathbf{S}'$ such that $(A, B) \leq I \leq (C, D)$ in $\mathcal{R}_c^*$. Let $(X, Y) \in \mathbf{S}'$. This implies $X \subseteq \mathcal{B}$ and $\mathcal{B}^c \subseteq Y$. Let $(A, B) \in \mathcal{R}_c^*$ such that (A, B) is a cover of (X, Y). That is, $(X, Y) \prec (A, B)$. Then $\mathcal{B}^c \subseteq Y \subseteq B$ which implies $(A, B) \in [(\emptyset, \mathcal{B}^c))$. Similarly, all the covers of (X, Y) are in $[(\emptyset, \mathcal{B}^c))$. Therefore from Theorem 4.5, we can conclude that $\mathcal{R}^*$ is a pasting of $((\mathcal{B}, U)]$ and $[(\emptyset, \mathcal{B}^c))$ together over $\mathbf{S}'$.

Conversely, assume $\mathcal{R}_c^*$ is a pasting of $((\mathcal{B}, U)]$ and $[(\emptyset, \mathcal{B}^c))$ together over $\mathbf{S}'$. Then $\mathcal{R}_c^* = ((\mathcal{B}, U)] \cup [(\emptyset, \mathcal{B}^c))$ which implies by Proposition 4.9, R is $*$-connected. Hence proved. ☐

The following corollary can be proposed, as it is shown in Proposition 3.3 that $((\mathcal{B}, U)] \cong \wp(U)^{\blacktriangle}$, $[(\emptyset, \mathcal{B}^c)) \cong \wp(U)^{\blacktriangledown}$ and $\mathbf{S}' \cong 2^{\mathcal{B}} \cong \mathbf{S}$ in Lemma 3.4.

Corollary 4.14. *Let R be any reflexive relation on a finite set U. Then $(\mathcal{R}_c^*, \leq)$ pastes the lattices $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$ together over $\boldsymbol{S}$ iff, R is $*$-connected.*

Proposition 4.15. *Let R be any reflexive relation on the set U. Then the following are equivalent*

(i) *$R(x) \subseteq R(y)$ or $R(y) \subseteq R(x)$, for all $x, y \in U$*
(ii) *$(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are linearly ordered (Chain).*
(iii) *$(\mathcal{R}_c^*, \leq)$ is linearly ordered (Chain).*

Proof. (i)⇒ (ii) Assume $R(x) \subseteq R(y)$ or $R(y) \subseteq R(x)$, for all $x, y \in U$ holds. Then this implies

$$\begin{aligned} & \{x\}^{\vartriangle} \subseteq \{y\}^{\vartriangle} \text{ or } \{y\}^{\vartriangle} \subseteq \{x\}^{\vartriangle}, \forall x, y \in U \\ \Rightarrow \; & \{x\}^{\vartriangle\blacktriangledown} \subseteq \{y\}^{\vartriangle\blacktriangledown} \text{ or } \{y\}^{\vartriangle\blacktriangledown} \subseteq \{x\}^{\vartriangle\blacktriangledown}, \forall x, y \in U \end{aligned}$$

All the elements in $\wp(U)^{\blacktriangledown}$ are comparable, since every element in $\wp(U)^{\blacktriangledown}$ is the union of $\{x\}^{\vartriangle\blacktriangledown}$. So $(\wp(U)^{\blacktriangledown}, \subseteq)$ is linearly ordered. Then by duality $(\wp(U)^{\blacktriangle}, \subseteq)$ is also linearly ordered.

(ii)⇒(iii) Assume (ii) holds. Suppose $(\mathcal{R}_c^*, \leq)$ is not linearly ordered, then $\exists (X, Y)$ and (V, W) in $\mathcal{R}_c^*$ such that $(X, Y) \not\leq (V, W)$ and also

$(V,W) \nleq (X,Y)$. This results in $X \nleq V$ and $V \nleq X$ or $Y \nleq W$ and $W \nleq Y$ which is a contradiction to the assumption that $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are linearly ordered. Therefore $(\mathcal{R}_c^*, \leq)$ is linearly ordered.

(iii)⇒ (i) Assume (iii) holds. Then all the elements in $\mathcal{R}_c^*$ are comparable. So, for any $x, y \in U$, $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle})$ and $(\{y\}^{\vartriangle\blacktriangledown}, \{y\}^{\vartriangle\blacktriangle})$ are comparable. This implies $(\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle}) \leq (\{y\}^{\vartriangle\blacktriangledown}, \{y\}^{\vartriangle\blacktriangle})$ or $(\{y\}^{\vartriangle\blacktriangledown}, \{y\}^{\vartriangle\blacktriangle}) \leq (\{x\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle})$

$$\begin{aligned}
&\Rightarrow \{x\}^{\vartriangle\blacktriangledown} \subseteq \{y\}^{\vartriangle\blacktriangledown}, \{x\}^{\vartriangle\blacktriangle} \subseteq \{y\}^{\vartriangle\blacktriangle} \text{ or} \{y\}^{\vartriangle\blacktriangledown} \subseteq \{x\}^{\vartriangle\blacktriangledown}, \{y\}^{\vartriangle\blacktriangle} \subseteq \{x\}^{\vartriangle\blacktriangle}\\
&\Rightarrow \{x\}^{\vartriangle\blacktriangledown} \subseteq \{y\}^{\vartriangle\blacktriangledown} \text{ or } \{y\}^{\vartriangle\blacktriangledown} \subseteq \{x\}^{\vartriangle\blacktriangledown}\\
&\Rightarrow \{x\}^{\vartriangle\blacktriangledown\vartriangle} \subseteq \{y\}^{\vartriangle\blacktriangledown\vartriangle} \text{ or } \{y\}^{\vartriangle\blacktriangledown\vartriangle} \subseteq \{x\}^{\vartriangle\blacktriangledown\vartriangle}\\
&\Rightarrow \{x\}^{\vartriangle} \subseteq \{y\}^{\vartriangle} \text{ or } \{y\}^{\vartriangle} \subseteq \{x\}^{\vartriangle}\\
&\Rightarrow R(x) \subseteq R(y) \text{ or } R(y) \subseteq R(x),
\end{aligned}$$

Hence proved. □

Let the condition on R in the above proposition be defined as follows.

Definition 4.16. A reflexive relation R on U is said to be L_*- connected, if for every $x, y \in U$, $R(x) \subseteq R(y)$ or $R(y) \subseteq R(x)$.

Proposition 4.17. *If R is $*$-connected on a finite set U, then $|\mathcal{R}_c^*| = 2|\wp(U)^{\blacktriangledown}| - 2^{|\mathcal{B}|}$.*

Proof. Assume R is $*$-connected. Since $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are dually isomorphic, $|\wp(U)^{\blacktriangledown}| = |\wp(U)^{\blacktriangle}|$ and $|\mathbf{S}| = 2^{|\mathcal{B}|}$. By Corollary 4.14, we can write $|\mathcal{R}_c^*| = |\wp(U)^{\blacktriangle}| + |\wp(U)^{\blacktriangledown}| - |\mathbf{S}| = 2|\wp(U)^{\blacktriangledown}| - 2^{|\mathcal{B}|}$. □

Corollary 4.18. *If R is $*$-connected on a finite set U and $\mathcal{B} = \emptyset$, then $|\mathcal{R}_c^*| = 2|\wp(U)^{\blacktriangledown}| - 1$.*

Corollary 4.19. *If R is L_*-connected on a finite set U such that $\mathcal{B} \neq \emptyset\emptyset$, then $|\mathcal{R}_c^*| = 2|\wp(U)^{\blacktriangledown}| - 2$.*

5. Conclusion

Studying the algebraic structure of $(\mathcal{R}_c^*, \leq)$ in the more generalised case of reflexive relation is helpful to identify the base structure of rough sets system for any relation. This work is useful for further study on rough sets in two approaches.

(i) It is well known that rough sets have mixed logic behavior. If the pattern of embedding of substructures of $(\mathcal{R}^*, \leq)$ in its Dedekind-MacNeille completion is identified, then it can be used to study the local logical behavior of the rough sets system based on various reflexive based relations. In this study, we have shown the pasting of rough approximation lattices defined by a reflexive relation over the common Boolean sublattice is order embedded in its Dedekind-MacNeille completion of the rough sets system. So, the local logical behaviour of $\mathcal{R}_c^*$ can be analyzed, as the rough sets in the portion $\mathbf{S}$ of $\mathcal{R}_c^*$ follow classical logic and rough sets above the portion $\mathbf{S}$, and below $\mathbf{S}$ in $(\mathcal{R}_c^*, \leq)$ follow the logic corresponding to the algebraic structures of $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ respectively.

(ii) The main task in studying the algebraic structure of $(\mathcal{R}_c^*, \leq)$ defined by any reflexive relation is to collect all the approximation pairs $(A^{\blacktriangledown}, A^{\blacktriangle})$ for $A \subseteq U$ and draw the Hasse diagram of it with the ordering $\leq$. If there is any simple procedure to draw the Hasse diagram of $(\mathcal{R}_c^*, \leq)$, then studying the algebraic structure of it will be easy. Here, we characterized the reflexive relation R, for which $(\mathcal{R}_c^*, \leq)$ defined by R is obtained by simply pasting its rough approximation lattices over a common Boolean sublattice. By this method, for such reflexive relations R, we can draw the Hasse diagram of $(\mathcal{R}_c^*, \leq)$ defined by R from its rough approximation lattices itself.

Acknowledgments

The author wishes to thank the anonymous reviewer for his/her valuable comments on the manuscript.

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Journal of Algebraic Systems

DEDEKIND–MACNEILLE COMPLETION OF THE ROUGH SETS SYSTEM AS PASTING OF ROUGH APPROXIMATION LATTICES

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مکمل ددکیند–مک‌نیل سیستم مجموعه‌های ناهموار به عنوان اتصال مشبکه‌های تقریب ناهموار

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در این مقاله، الگوی جانشانی مشبکه‌های تقریب ناهموار که توسط یک رابطه بازتابی تعریف شده‌اند، در مکمل ددکیند–مک‌نیل سیستم مجموعه‌های ناهموار آن‌ها مورد بررسی قرار گرفته است. رابطه بازتابی R که برای آن مکمل ددکیند–مک‌نیل سیستم مجموعه‌های ناهموار تعریف شده توسط R، اتصال مشبکه‌های تقریب ناهموار آن است، مشخص می‌شود. برخی از خواص مکمل ددکیند–مک نیل سیستم مجموعه‌های ناهموار تعریف شده توسط یک رابطه بازتابی R نیز مورد بحث قرار می‌گیرند.

کلمات کلیدی: سیستم مجموعه‌های ناهموار، تقریب‌های ناهموار، اجتماع–چگال، اتصال مشبکه‌ها.