Journal of Algebraic Systems
Vol. 2, No. 2, (2014), pp 125-135

COHEN-MACAULAY HOMOLOGICAL DIMENSIONS WITH RESPECT TO AMALGAMATED DUPLICATION

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Abstract. In this paper we use "ring changed" Gorenstein homological dimensions to define Cohen-Macaulay injective, projective and flat dimensions. To do this we use the amalgamated duplication of the base ring with semi-dualizing ideals. Finiteness of these new dimensions characterize Cohen-Macaulay rings with dualizing ideals.

1. Introduction

Let A be a commutative ring with identity and E an A-module. In 1956, Nagata introduced the trivial extension of A along E denoted by $A \ltimes E$ (cf. [13]). As an A-module, $A \ltimes E$ is just the direct sum of A and M. The multiplication is defined by $(a, x)(b, y) = (ab, ay + bx)$ for all $a, b \in A$ and $x, y \in E$. In [10], Holm and Jorgensen introduced the Cohen-Macaulay injective, projective and flat dimension of a complex. In fact if A is a ring with a semi-dualizing module C, then one can consider the trivial extension ring $A \ltimes C$, and if M is a complex of A-modules, then we can consider M as a complex of $(A \ltimes C)$-modules and take the Gorenstein homological dimensions of M over $(A \ltimes C)$. Then the infima of these over all semi-dualizing modules C define the Cohen-Macaulay dimensions of M as follows:

Definition 1.1. ([10, Definition 2.3]) Let M and N be complexes of A-modules such that $\mathrm{H}_i(M) = 0$ for $i \gg 0$ and $\mathrm{H}_i(N) = 0$ for $\ll 0$.

MSC(2010): Primary: 13D05; Secondary: 13D09, 13C14.
Keywords: Semi-dualizing ideal, Amalgamated duplication, Gorenstein homological dimension, Cohen-Macaulay homological dimension.
Received: 24 July 2014, Revised: 16 December 2014.

The Cohen-Macaulay injective, projective, and flat dimensions of M and N over A are:

$$\mathrm{CMid}_A M = \inf\{\mathrm{Gid}_{A\ltimes C} M \mid C \text{ is a semi-dualizing module}\},$$

$$\mathrm{CMpd}_A N = \inf\{\mathrm{Gpd}_{A\ltimes C} N \mid C \text{ is a semi-dualizing module}\},$$

$$\mathrm{CMfd}_A N = \inf\{\mathrm{Gfd}_{A\ltimes C} N \mid C \text{ is a semi-dualizing module}\}.$$

The finiteness of these Cohen-Macaulay dimensions are equivalent that, the ring A is Cohen-Macaulay with a dualizing module. Recall from [3] that, a finitely generated A-module C is called a *semi-dualizing module* for A if $A \longrightarrow \mathbf{R}\operatorname{Hom}_A(C,C)$ is an isomorphism in the derived category $D(A)$ of A. Equivalently a finitely generated A-module C is called a semi-dualizing A-module if $\operatorname{Ext}^i_A(C,C) = 0$ for all $i \geqslant 0$ and $\operatorname{Hom}_A(C,C) \cong A$. Recall that A has at least one semi-dualizing module e.g. $C = A$ itself.

Recently D'anna and Fontana have introduced a new construction, called the *amalgamated duplication* of a ring A along an ideal $I \subset A$, denoted by $A \bowtie I$. When $I^2 = 0$ the new construction $A \bowtie I$ coincides with the notion of trivial extension $A \ltimes I$. The main properties of the amalgamated duplication $A \bowtie I$ have discussed more in detail in [6] and [7].

In this paper we consider $A \bowtie I$ construction when I is a semi-dualizing ideal of A and take the ring changed Gorenstein homological dimension of a A-complex over $A \bowtie I$. Then we define the Cohen-Macaulay injective, projective, and flat dimensions denoted by $\mathrm{CM}_{\bowtie}\mathrm{id}_A$, $\mathrm{CM}_{\bowtie}\mathrm{pd}_A$, and $\mathrm{CM}_{\bowtie}\mathrm{fd}_A$ respectively (see Definition 2.1). Our main result is Theorem 2.8 which gives a characterization of Cohen-Macaulay rings admitting dualizing ideals by finiteness of the new Cohen-Macaulay dimensions.

Next, we deal with some applications of a general construction, introduced in [7], called amalgamated duplication of a ring along an ideal.

We end this section with a description of the construction of D'Anna and Fontana's amalgameted duplication. Let A be a commutative ring with unit element 1 and let I be an ideal of A. Set

$$A \bowtie I = \{(a,b) | a,b \in A, b-a \in I\}.$$

It is easy to check that $A \bowtie I$ is a subring of $A \times A$ with unit element $(1,1)$ (with the usual componentwise operations) and that $A \bowtie I =$

$\{(a, a+i)|a \in A, i \in I\}$. There are ring homomorphisms

$$\begin{aligned} A \longrightarrow & A \bowtie I \longrightarrow A, \\ a \longmapsto & (a,a), \\ & (a,c) \longmapsto a. \end{aligned}$$

These homomorphisms allow us to view any A-module as an $A \bowtie I$-module, and any $A \bowtie I$-module as an A-module.

Throughout this paper A will denoted for a commutative and Noetherian ring with unit element 1.

2. Cohen-Macaulay dimensions

In this section, we define the Cohen-Macaulay dimensions with respect to amalgamated duplication along semi-dualizing ideals, and then identify when these quantities are finite. The key definition is the following. Recall that a semi-dualizing ideal, is an ideal of A which is a semi-dualizing A-module. Note that every ideal of A isomorphic to A is a semi-dualizing ideal. In particular A is a semi-dualizing ideal of A.

Definition 2.1. Let M and N be complexes of A-modules such that $\mathrm{H}_i(M) = 0$ for $i \gg 0$ and $\mathrm{H}_i(N) = 0$ for $\ll 0$.

The Cohen-Macaulay injective, projective, and flat dimensions with respect to amalgamated duplication, of M and N over A are:

$$\mathrm{CM}_{\bowtie}\mathrm{id}_A\, M = \inf\{\mathrm{Gid}_{A\bowtie I}\, M \mid I \text{ is a semi-dualizing ideal}\},$$

$$\mathrm{CM}_{\bowtie}\mathrm{pd}_A\, N = \inf\{\mathrm{Gpd}_{A\bowtie I}\, N \mid I \text{ is a semi-dualizing ideal}\},$$

$$\mathrm{CM}_{\bowtie}\mathrm{fd}_A\, N = \inf\{\mathrm{Gfd}_{A\bowtie I}\, N \mid I \text{ is a semi-dualizing ideal}\},$$

where Gid, Gpd, and Gfd denote the Gorenstein injective, projective, and flat dimensions (see [2]).

The main result of this paper is Theorem 2.8. For the proof of the theorem we need the following proposition and lemmas. Recall the definition of the Bass and Auslander classes with respect to the semi-dualizing ideal I, from [3, Definition. 4.1]. Let I be a semi-dualizing ideal of A. The I-Bass class of A, $\mathcal{B}_I(A)$, is the full subcategory of the derived category of A, defined by specifying their objects as follows: Y belongs to $\mathcal{B}_I(A)$ if and only if Y and $\mathbf{R}\operatorname{Hom}_A(I, Y)$ are homologically bounded, and the canonical map $\xi_Y^I : I \otimes_A^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_A(I, Y) \to Y$ is an isomorphism. The I-Auslander class of A, $\mathcal{A}_I(A)$, is the full subcategory of the derived category of A, defined by specifying their objects as follows: X belongs to $\mathcal{A}_I(A)$ if and only if X and $I \otimes^{\mathbf{L}}_A X$ are homologically bounded, and the canonical map $\gamma^I_X : X \to \mathbf{R}\operatorname{Hom}_A(I, I \otimes^{\mathbf{L}}_A X)$ is an isomorphism. By [3, Proposition 4.4] all complexes of finite injective (resp. flat) dimension is belong to $\mathcal{B}_I(A)$ (resp. $\mathcal{A}_I(A)$).

Proposition 2.2. *Let I be a semi-dualizing ideal of A and let M be an A-module which is Gorenstein injective over $A \bowtie I$. Then there exists a short exact sequence of A-modules,*

$$0 \longrightarrow M' \longrightarrow \operatorname{Hom}_A(I, E) \longrightarrow M \longrightarrow 0,$$

where E is an injective A-module, and M' is Gorenstein injective over $A \bowtie I$, which stays exact if one applies to it the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(I, J), -)$ for any injective A-module J.

Proof. The argument is the same as proof of [10, Lemma 4.1] with some changes. Since M is a Gorenstein injective $A \bowtie I$-module, there exists a short exact sequence of $A \bowtie I$-modules,

$$0 \longrightarrow N \longrightarrow K \longrightarrow M \longrightarrow 0,$$

where K is injective, and N is Gorenstein injective $A \bowtie I$-modules, which stays exact if one applies to it the functor $\operatorname{Hom}_{A\bowtie I}(L, -)$ for any injective $A \bowtie I$-module L. By [15, Lemma 3.7(i)], for any injective A-module J, $\operatorname{Hom}_A(A \bowtie I, J)$ is injective $A \bowtie I$-module. Thus the short exact sequence stays exact if one applies to it the functor $\operatorname{Hom}_{A\bowtie I}(\operatorname{Hom}_A(A \bowtie I, J), -)$. On the other hand by [15, Lemma 3.1(v)] there is the following equivalence of functors:

$$\operatorname{Hom}_{A\bowtie I}(\operatorname{Hom}_A(A \bowtie I, J), -) \cong \operatorname{Hom}_A(\operatorname{Hom}_A(I, J), -).$$

Therefore the above short exact sequence, stays exact when one applies to it the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(I, J), -)$. By [15, Lemma 3.7(ii)] the injective $A \bowtie I$-module K is a direct summand in $\operatorname{Hom}_A(A \bowtie I, E)$ for some injective A-module E. If $K \oplus K' \cong \operatorname{Hom}_A(A \bowtie I, E)$, then by adding K' to both the first and second modules in the above short exact sequence, we may assume that, the sequence has the form:

$$0 \longrightarrow N \longrightarrow \operatorname{Hom}_A(A \bowtie I, E) \longrightarrow M \longrightarrow 0.$$

The $A \bowtie I$-module structure of $\operatorname{Hom}_A(A \bowtie I, E)$ comes from the first variable. For any $a \in A$, $i \in I$ and $(\alpha, \gamma) \in \operatorname{Hom}_A(A \bowtie I, E)$ we have:

$$\begin{pmatrix} a \\ i \end{pmatrix} . (\alpha, \gamma) = (a\alpha + \chi_{\gamma(i)}, a\gamma + \chi'_{\gamma(i)})$$

where $\chi_{\gamma(i)}$ is the homomorphism $A \to E$ given by $a \mapsto a.\gamma(i)$ and $\chi'_{\gamma(i)}$ is the homomorphism $I \to E$ given by $j \mapsto j.\gamma(i)$.

When we view M as an $A \bowtie I$-module, it is annihilated by the ideal $\mathfrak{p} = \{(0,i) | i \in I\}$ (See [7, Lemma 2.4]). Hence

$$0 = \begin{pmatrix} 0 \\ i \end{pmatrix} .\eta\,(\alpha,\gamma) = \eta(\begin{pmatrix} 0 \\ i \end{pmatrix} .(\alpha,\gamma)) = \eta(\chi_{\gamma(i)}, \chi'_{\gamma(i)}). \quad (*)$$

Let $\alpha : A \to E$ be a homomorphism. Therefore $\alpha(a) = a.\alpha(1)$ for any $a \in A$. Note that there exists a surjection $F \to \mathrm{Hom}_A(I,E)$ with F free, and hence a surjection $I \otimes_A F \to I \otimes \mathrm{Hom}_A(I,E)$. On the other hand as E is injective, it belongs to the Bass class, $\mathcal{B}_I(A)$, and therefore $I \otimes \mathrm{Hom}_A(I,E) \cong E$ (See [3, Proposion 4.4 and Obseversion 4.10]). So there is a surjection $I \otimes_A F \to E$. Since $I \otimes_A F$ is a direct sum of copies of I, this means that there exist the homomorphisms $\gamma_1, \dots, \gamma_t : I \to E$ and the element $i_1, \dots, i_t \in I$, such that $\alpha(1) = \gamma_1(i_1) + \cdots + \gamma_t(i_t)$. Hence $\alpha : A \to E$ is equal to $\chi_{\gamma_1(i_1)+\cdots+\gamma_t(i_t)} = \chi_{\gamma_1(i_1)} + \cdots + \chi_{\gamma_t(i_t)}$. Therefore $(*)$ implies that $\eta(\alpha, \alpha|_I) = 0$ for every $\alpha : A \to I$. Now we have the following short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(A,E) \xrightarrow{f} \mathrm{Hom}_A(A \bowtie I, E) \xrightarrow{g} \mathrm{Hom}_A(I,E) \longrightarrow 0,$$

where $f(\alpha) = (\alpha, \alpha|I)$ and $g(\alpha,\gamma) = \gamma - \alpha|I$, for any $\alpha \in \mathrm{Hom}_A(A,E)$ and $\gamma \in \mathrm{Hom}_A(I,E)$. Hence we can construct a commutative diagram of $A \bowtie I$-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & \mathrm{Hom}_A(A \bowtie I, E) \arrow[r, "\eta"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & M' \arrow[r] & \mathrm{Hom}_A(I,E) \arrow[r, "\varphi"'] & M \arrow[r] & 0.
\end{tikzcd}$$

For this let $\gamma \in \mathrm{Hom}_A(I,E)$ then there exists $(\alpha_1,\gamma_1) \in \mathrm{Hom}_A(A \bowtie I, E)$ such that $g(\alpha_1,\gamma_1) = \gamma_1 - \alpha_1|I = \gamma$. Hence we can define $\varphi(\gamma) = \eta(\alpha_1,\gamma_1)$. It is clear that φ is well defined. Indeed let $\gamma = \gamma'$ then there exist $(\alpha_1,\gamma_1), (\alpha'_1,\gamma'_1) \in \mathrm{Hom}_A(A \bowtie I, E)$ such that $g(\alpha_1,\gamma_1) = g(\alpha'_1,\gamma'_1)$. Therefore $\gamma_1 - \alpha_1|I = \gamma'_1 - \alpha'_1|I$ implying $\eta(0, \gamma_1 - \alpha_1|I) = \eta(0, \gamma'_1 - \alpha'_1|I)$. So by $(*)$,

$$\varphi(\gamma) = \eta(\alpha_1,\gamma_1) = \eta(\alpha'_1,\gamma'_1) = \varphi(\gamma').$$

Also φ is surjective since η and g are. The exact sequence,

$$0 \longrightarrow M' \longrightarrow \mathrm{Hom}_A(I,E) \longrightarrow M \longrightarrow 0,$$

has the properties claimed in the proposition. To see so, applying the snake lemma to the above diagram, we find the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_A(A,E) \longrightarrow N \longrightarrow M' \longrightarrow 0.$$

By [15, Theorem 3.8(ii)], $\operatorname{Hom}_A(A, E)$ is Gorenstein injective over $A \bowtie I$, also N is Gorenstein injective over $A \bowtie I$ by construction. Therefore M' is Gorenstein injective over $A \bowtie I$ since the class of Gorenstein injective modules is injectively resolving by [9, Theorem. 2.7]. Finally by construction, the upper sequence in the diagram stays exact if one applies to it the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(I, J), -)$ for any injective A-module J. So the lower sequence in the diagram stays exact when one applies the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(I, J), -)$ to it. $\square$

As the same method as [10, Lemma 4.2] we can use Proposition 2.2 and [15, Theorem 3.8(ii)] to prove the following lemma, so we omit its proof. By Σ is denoted suspension of complexes in the derived category. The injective (resp. flat) dimension of a A-complex N is denoted by $\operatorname{id}_A N$ (resp. $\operatorname{fd}_A N$).

Lemma 2.3. *Let I be a semi-dualizing module for A. Let M be a complex in $\mathcal{A}_I(A)$ which has non-zero homology and satisfies that $\operatorname{Gid}_{A\bowtie I} M < \infty$. Write $s = \sup\{i | \operatorname{H}_i(M) \neq 0\}$. Then there is a distinguished triangle in $\mathcal{D}(A)$,*

$$\Sigma^s H \to Y \to M \to,$$

where H is an A-module which is Gorenstein injective over $A \bowtie I$ and where $\operatorname{id}_A(I \otimes^{\mathbf{L}}_A Y) \leqslant \operatorname{Gid}_{A\bowtie I} M$.

Lemma 2.4. *Let I be a semi-dualizing ideal of A. Let M be a complex in $\mathcal{A}_I(A)$ with non-zero homology. Set $s = \sup\{i | \operatorname{H}_i(M) \neq 0\}$ and suppose that M satisfies*

$$\operatorname{Ext}^{s+1}_A(M, H) = 0$$

for each A-module H which is Gorenstein injective over $A \bowtie I$. Then

$$\operatorname{id}_A(I \otimes^{\mathbf{L}}_A M) = \operatorname{Gid}_{A\bowtie I} M.$$

Proof. The proof is similar to the proof of [10, Lemma 4.3]. Just use Lemma 2.3 instead of [10, Lemma 4.2] and [15, Theorem 3.8(ii)] instead of [10, Lemma 3.3(ii)]. $\square$

The following Lemma shows that there exist some complexes which lemma 2.4 applies. The proof is similar to the proof of [10, Lemma 4.4], therefore we omit it.

Lemma 2.5. *Let I be a semi-dualizing ideal of A. Let M be a complex of A-modules which has non-zero homology and satisfies that $\operatorname{H}_i(M) = 0$ for $i \ll 0$ and that projective dimension of M as A-module is finite. Write $s = \sup\{i | \operatorname{H}_i(M) \neq 0\}$. Let H be an A-module which is*

Gorenstein injective over $A \bowtie I$. Then

$$\operatorname{Ext}_A^{s+1}(M,H)=0.$$

Recall from [2, (A.6.3)] that when A is local with residue class field k, and M is a complex of A-modules such that $\mathrm{H}_i(M)=0$ for $i \ll 0$, the *width* of M is define as

$$\operatorname{width}_A M = \inf\{i|\ \mathrm{H}_i(M \otimes_A^{\mathbf{L}} k) \neq 0\}.$$

Recall also from [2, (A.6.3)] that when A is local with residue class field k and M is a complex of A-modules such that $\mathrm{H}_i(M)=0$ for $i \gg 0$, the *depth* of a complex of M, is define as

$$\operatorname{depth}_A M = -\sup\{i|\ \mathrm{H}_i(\mathbf{R}\operatorname{Hom}_A(k,M)) \neq 0\}.$$

During results 2.6-2.11 we barrow methods of [10] to establish analogues of trivial extension properties appeared in [10], for amalgameted duplication.

Lemma 2.6. *Assume that A is a local ring with residue class field k and I is a semi-dualizing ideal of A. Let M be a complex of A-modules which has non-zero homology and satisfies that $\mathrm{H}_i(M)=0$ for $i \ll 0$ and that $\operatorname{fd}_A M < \infty$. Then*

$$\operatorname{id}_A I \leq \operatorname{Gid}_{A\bowtie I} M + \operatorname{width}_A M.$$

Proof. If $\operatorname{width}_A M = \infty$, then there is nothing to proof. On the other hand $\mathrm{H}_i(M)=0$ for $i \ll 0$ implies that $\mathrm{H}_i(M \otimes_A^{\mathbf{L}} k)=0$ for $i \ll 0$, whence $\operatorname{width}_A M > -\infty$. Thus we may assume that $-\infty < \operatorname{width}_A M < \infty$. The condition $\operatorname{fd}_A M < \infty$ implies $\operatorname{pd}_A M < \infty$, since $\dim A < \infty$ (cf. [12, Proposition 6]). Let $s = \sup\{i|\ \mathrm{H}_i(M) \neq 0\}$ and H is any A-module which is Gorenstein injective over $A \bowtie I$. Then $\operatorname{Ext}_A^{s+1}(M,I)=0$, by Lemma 2.5. Note that $M \in \mathcal{A}_I(A)$ hence by Lemma 2.4, $\operatorname{id}_A(I \otimes_A^{\mathbf{L}} M) = \operatorname{Gid}_{A\bowtie I} M$. Now we have the following computations

$$\begin{aligned}
\operatorname{Gid}_{A\bowtie I} M &= \operatorname{id}_A(I \otimes_A^{\mathbf{L}} M)\\
&\geqslant -\inf\{i|H_i(\mathbf{R}\operatorname{Hom}_A(k, I \otimes_A^{\mathbf{L}} M)) \neq 0\}\\
&= -\inf\{i|H_i(\mathbf{R}\operatorname{Hom}(k,I) \otimes_A^{\mathbf{L}} M) \neq 0\}\\
&= -\inf\{i|H_i(\mathbf{R}\operatorname{Hom}(k,I)) \neq 0\} - \inf\{i|H_i(M \otimes_A^{\mathbf{L}} k) \neq 0\}\\
&= \operatorname{id}_A I - \operatorname{width}_A M,
\end{aligned}$$

where the first inequality is from [2, A.5.2], the second equality is by Tensor-evaluation ([2, A.4.23]), since $\operatorname{fd}_A M < \infty$, the third equality is by [2, A.7.9.2] and the last equality is by [2, A.5.7.4] and definition of width. ∎

Lemma 2.7. *Assume that A is a local ring with residue class field k and I is a semi-dualizing ideal of A. Let N be a complex of A-modules which has non-zero homology and satisfies that $\mathrm{H}_i(N) = 0$ for $i \gg 0$ and that $\mathrm{id}_A N < \infty$. Then*

$$\mathrm{id}_A I \leq \mathrm{Gfd}_{A\bowtie I} N + \mathrm{depth}_A N.$$

Proof. Let $E = E(k)$ be the injective envelope of k. Then it is well known that $\mathrm{fd}_A \mathrm{Hom}_A(N, E) = \mathrm{id}_A N$ and $\mathrm{width}_A \mathrm{Hom}_A(N, E) = \mathrm{depth}_A N$.
On the other hand by [15, Lemma 3.7], $\mathrm{Hom}_A(A \bowtie I, E)$ is a faithfully injective $A \bowtie I$-module. Hence $\mathrm{Gid}_{A\bowtie I} \mathrm{Hom}_{A\bowtie I}(N, \mathrm{Hom}_A(A \bowtie I, E)) = \mathrm{Gfd}_{A\bowtie I} N$ follows from [2, Theorem 6.4.2]. Thus by associativity, $\mathrm{Gid}_{A\bowtie I} \mathrm{Hom}_A(N, E) = \mathrm{Gfd}_{A\bowtie I} N$. Now the desired inequality holds by Lemma 2.6 applying to $M = \mathrm{Hom}_A(N, E)$. □

Now we are ready to prove the main result of this paper. The flat dimension of a A-module M is denoted by $\mathrm{fd}_A M$.

Theorem 2.8. *Assume that the ring A is local with residue class field k. The following are equivalent.*

1. *A is a Cohen-Macaulay ring with a dualizing ideal.*
2. *$\mathrm{CM}_{\bowtie}\mathrm{id}_A M < \infty$ holds when M is any complex of A-modules with bounded homology.*
3. *There is a complex M of A-modules with bounded homology, $\mathrm{CM}_{\bowtie}\mathrm{id}_A M < \infty$, $\mathrm{fd}_A M < \infty$ and $\mathrm{width}_A M < \infty$.*
4. *$\mathrm{CM}_{\bowtie}\mathrm{id}_A k < \infty$.*
5. *$\mathrm{CM}_{\bowtie}\mathrm{pd}_A M < \infty$ holds when M is any complex of A-modules with bounded homology.*
6. *There is a complex M of A-modules with bounded homology, $\mathrm{CM}_{\bowtie}\mathrm{pd}_A M < \infty$, $\mathrm{id}_A M < \infty$ and $\mathrm{depth}_A M < \infty$.*
7. *$\mathrm{CM}_{\bowtie}\mathrm{pd}_A k < \infty$.*
8. *$\mathrm{CM}_{\bowtie}\mathrm{fd}_A M < \infty$ holds when M is any complex of A-modules with bounded homology.*
9. *There is a complex M of A-modules with bounded homology, $\mathrm{CM}_{\bowtie}\mathrm{fd}_A M < \infty$, $\mathrm{id}_A M < \infty$ and $\mathrm{depth}_A M < \infty$.*
10. *$\mathrm{CM}_{\bowtie}\mathrm{fd}_A k < \infty$.*

Proof. (1) ⇒ (2) Assume that A is a Cohen-Macaulay ring with a dualizing ideal I. Therefore by [6, Theorem 11] or [1, Corollary 3.4] $A \bowtie I$ is a Gorenstein ring. Thus by [2, Theorem 6.2.7] $\mathrm{Gid}_{A\bowtie I} M < \infty$. Note that M is a complex with bounded homology as $A \bowtie I$-module. But I is in particular a semi-dualizing ideal, so by Definition 2.1 we get $\mathrm{CM}_{\bowtie}\mathrm{id}_A M < \infty$.

(2) ⇒ (3) and (2) ⇒ (4) are trivial.

(3) $\Rightarrow$ (1) Assume that M is a complex of A-modules with bounded homology such that $\mathrm{CM}_{\bowtie}\mathrm{id}_A\, M < \infty$, $\mathrm{fd}_A\, M < \infty$, and $\mathrm{width}_A\, M < \infty$. So that there is a semi-dualizing ideal I of A such that $\mathrm{Gid}_{A\bowtie I}\, M < \infty$. Therefore by Lemma 2.6 we have $\mathrm{id}_A\, I < \infty$, which implies that A is a Cohen-Macaulay ring with dualizing ideal I.

(4) $\Rightarrow$ (1) Assume that $\mathrm{CM}_{\bowtie}\mathrm{id}_A\, k < \infty$. Then by Definition 2.1, A has a semi-dualizing ideal I with $\mathrm{Gid}_{A\bowtie I}\, k < \infty$. Set $E := E_{A\bowtie I}(k)$ be the injective envelope of k as $A \bowtie I$-module. Therefore $\mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(E, k)$ has bounded homology by [9, Theorem 2.22]. Using the isomorphism $k \cong \mathrm{Hom}_{A\bowtie I}(k, E)$ and the adjunction isomorphism, we have:

$$\begin{aligned}
\mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(E, k) &\cong \mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(E, \mathrm{Hom}_{A\bowtie I}(k, E)) \\
&\cong \mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(E \otimes_{A\bowtie I} k, E) \\
&\cong \mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(k, \mathrm{Hom}_{A\bowtie I}(E, E)) \\
&\cong \mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(k, \widehat{A \bowtie I}) \\
&\cong \mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(k, A \bowtie I) \otimes_{A\bowtie I} \widehat{A \bowtie I}.
\end{aligned}$$

Since $\widehat{A \bowtie I}$ is faithfully flat over $A \bowtie I$, it follows that $\mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(k, A \bowtie I)$ has also bounded homology. Thus $A \bowtie I$ is a Gorenstein ring. Hence by [6, Theorem 11] A is a Cohen-Macaulay ring with dualizing ideal I.

(1) $\Rightarrow$ (5) the technique is similar to (1) $\Rightarrow$ (2).

(5) $\Rightarrow$ (6) and (5) $\Rightarrow$ (7) are trivial.

(6) $\Rightarrow$ (1) the technique is similar to (3) $\Rightarrow$ (1) just note that A has finite Krull dimension since it is a local ring. Hence by [14] and [12] every flat module has finite projective dimension. Therefore by [4, Proposition 3.7] $\mathrm{Gpd}_{A\bowtie I}\, M < \infty$ implies that $\mathrm{Gfd}_{A\bowtie I}\, M < \infty$. And use Lemma 2.7 instead of Lemma 2.6.

(7) $\Rightarrow$ (1) Assume that $\mathrm{CM}_{\bowtie}\mathrm{pd}_A\, k < \infty$ then A has a semi-dualizing ideal I with

$$\mathrm{Gpd}_{A\bowtie I}\, k < \infty.$$

Therefore $\mathbf{R}\,\mathrm{Hom}_{A\bowtie I}(k, A \bowtie I)$ has bounded homology by [9, Theorem 2.20]. So that $A \bowtie I$ is a Gorenstein ring. Hence by [6, Theorem 11] A is a Cohen-Macaulay ring with dualizing ideal I.

With similar proofs one can see easily that (1), (8), (9) and (10) are equivalent. □

In [8], Gerko introduced the notion of Cohen-Macaulay dimension of finitely generated A-modules M denoted by $\text{CM-}\dim_A M$. It is known that a local ring A with residue class field k is Cohen-Macaulay if and only if $\text{CM-}\dim_A k < \infty$, by [8, Theorem 3.9].

Proposition 2.9. *Assume that the ring A is local and let M be a finitely generated A-module. Then*

$$CM\text{-}\dim_A M \leq \mathrm{CM}_{\bowtie}\mathrm{pd}_A M,$$

and if $\mathrm{CM}_{\bowtie}\mathrm{pd}_A M$ *is finite then equality holds.*

Proof. Take a semi-dualizing ideal I such that $\mathrm{CM}_{\bowtie}\mathrm{pd}_A M = \mathrm{Gpd}_{A\bowtie I} M$. On the other hand combining [8, proof of Theorem 3.7] with [8, Definition 3.2'] shows that

$$\mathrm{CM}\text{-}\dim_A M \leq \mathrm{Gpd}_{A\bowtie I} M.$$

So $\mathrm{Gpd}_{A\bowtie I} M < \infty$ implies that $\mathrm{CM}\text{-}\dim_A M < \infty$ and hence

$$\mathrm{CM}\text{-}\dim_A M = \operatorname{depth} A - \operatorname{depth}_A M$$

by [8, Theorem 3.8]. Now $\mathrm{Gpd}_{A\bowtie I} M = \mathrm{G}\text{-}\dim_I M$ by [11, Proposition 3.1], where $\mathrm{G}\text{-}\dim_I M$ is the Gorenstein dimension with respect to the semi-dualizing ideal I introduced in [3, Definition 3.11]. So $\mathrm{G}\text{-}\dim_I M$ is finite and hence

$$\mathrm{G}\text{-}\dim_I M = \operatorname{depth} A - \operatorname{depth}_A M$$

by [3, Theorem 3.14]. Combining the last four equations shows that $\mathrm{CM}\text{-}\dim_A M = \mathrm{CM}_{\bowtie}\mathrm{pd}_A M$ as desired. □

In the proof of the last result we saw that $\mathrm{CM}_{\bowtie}\mathrm{pd}_A M$ enjoys the Auslander-Buchsbaum formula.

Proposition 2.10. *Assume that the ring A is local and let M be a finitely generated A-module. If* $\mathrm{CM}_{\bowtie}\mathrm{pd}_A M < \infty$*, then*

$$\mathrm{CM}_{\bowtie}\mathrm{pd}_A M = \operatorname{depth} A - \operatorname{depth}_A M.$$

Dually we have the Bass formula:

Proposition 2.11. *Assume that the ring A is local and let $N \neq 0$ be a finitely generated A-module. If* $\mathrm{CM}_{\bowtie}\mathrm{id}_A M < \infty$*, then*

$$\mathrm{CM}_{\bowtie}\mathrm{id}_A M = \operatorname{depth} A.$$

Proof. Take a semi-dualizing ideal I such that $\mathrm{CM}_{\bowtie}\mathrm{id}_A M = \mathrm{Gid}_{A\bowtie I} M$. By [5, Theorem 2.2] finiteness of $\mathrm{Gid}_{A\bowtie I} M$ implies that

$$\mathrm{Gid}_{A\bowtie I} M = \operatorname{depth}_{A\bowtie I} A \bowtie I.$$

Finally note that we have

$$\operatorname{depth}_{A\bowtie I} A \bowtie I = \operatorname{depth} A.$$

□

Acknowledgments

The author wish to thank the referees for their careful reading and usefull suggestions.

References

1. A. Bagheri, M. Salimi, E. Tavasoli and S. Yassemi, A construction of Quasi-Gorenstein rings, *J. Algebra Appl.* **11**(1) (2012), 1250013, (9 pages).
2. L. W. Christensen, Gorenstein dimensions, *Lecture Notes in Mathematics* **1747** Springer-Verlag, Berlin, 2000.
3. L. W. Christensen, Semi-dualizing complex and their Auslander categories, *Trans. Amer. Math. Soc.* **353** (2001), 1839-1883.
4. L. W. Christensen, A. Frankild and H. Holm, On Gorenstein Projective, Injective and Flat dimensions - A Functorial description with applications, *J. Algebra,* **302** (2006), 231-279.
5. L. W. Christensen and S. Sather-Wagstaff, Transfer of Gorenstein dimensions along ring homomorphisms, *J. Pure Appl. Algebra,* **214** (2010), 982-989.
6. M. D'Anna, A consrtruction of Gorenstein ring, *J. Algebra,* **306** (2006), 509-519.
7. M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.* **6**(3) (2007), 443-459.
8. A. A. Gerko, On homological dimensions, *Sb. Math.* **192** (2001), 1165-1179.
9. H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra,* **189** (2004), 167-193.
10. H. Holm and P. Jorgenson, Cohen-Macaulay homological dimensions, *Rend. Sem. Math. Univ. Padova,* **117** (2007), 87-112.
11. H. Holm and P. Jorgenson, Semi-dualizing modules and related Gorenstein homological dimensions, *J. Pure Appl. Algebra,* **205** (2006), 423-445.
12. C. U. Jensen, On the vanishing of $\varprojlim^{(i)}$, *J. Algebra*, **15** (1970), 151-166.
13. M. Nagata, Local rings, *Interscience, New York* 1962.
14. M. Raynaud and L. Gruson, Critères de platitude et de projectivité. Techniques de "platification" d'un module, *Invent. Math.* **13** (1971), 1-89.
15. M. Salimi, E. Tavasoli and S. Yassemi, The amalgamated duplication of a ring along a semidualizing ideal, *Rend. Sem. Mat. Univ. Padova,* to appear.

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