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AN INTEGRAL DEPENDENCE IN MODULES OVER COMMUTATIVE RINGS

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Abstract. In this paper, a generalization of the integral dependence of rings on modules is given. The stability of the integral closure with respect to various module theoretic constructions is also studied. Moreover, the notion of integral extension of a module is introduced, and the Lying over, Going up and Going down theorems for modules are proved.

1. Introduction

Throughout this paper, all rings are commutative with identity, and all modules are unital. In the commutative ring theory, the integral element is defined, and its properties are discussed. Also the Lying over, Going up and Going down theorems are stated in many texts such as [3]. Let $R \subseteq R'$ be the rings. $\alpha \in R'$ is the integral over R, if there exists a monic polynomial $f(x) \in R[x]$, such that $f(\alpha) = 0$ [3]. In this paper, we introduce the notion of integral elements in a module (Definition 2.1). If $R \subseteq R' \subseteq K$ are the rings, and K is a quotient field of R, then $\alpha \in R'$ is the integral over a ring R, if and only if $\alpha.1_R$ is the integral over R when R is regarded as an R-module. Let M' be an R-module, and S be the set of regular elements of R. Then

$$T_{M'} = \{t \in S : tm' = 0, \text{ for some } m' \in M' \text{ implies that } m' = 0\}$$

is a multiplicative closed subset of R. As Naoum and Al-Alwan have stated [5], we say that $yn \in M$, as long as there exists an element m

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in M such that $tm = rn$, where $y = r/t \in T_M^{-1}R$, and $n \in M - \{0\}$. For any submodule N of an R-module M, we define $(N : M) = \{r \in R | rM \subseteq N\}$. A submodule P of M is called prime, if $P \neq M$; and for $r \in R$, $m \in M$ and $rm \in P$, we have $m \in P$ or $r \in (P : M)$. It is easy to show that if P is a prime submodule of an R-module M, then $(P : M)$ is a prime ideal of R [4]. The set of all prime submodules is denoted by $Spec(M)$.

This paper has been organized as follows: In section 2, we discuss the concept of an integral element over a module, a generalization of the concept of an integral element over a ring. In Theorem 2.3, we obtain the equivalent characterizations for the integral elements. We show, in Lemma 2.5 that if $M \subseteq M'$ are R-modules, and $y \in T_{M'}^{-1}R$, then $\overline{M}_{M'}^{y}$ is an R-module. Section 3 is devoted to introducing the concept of integral extension and the integrally closed module. In [1], Alkan and Tiras have defined the integrally closed module. Here, we define the notion of integrally closed modules in connection with the integral elements. Then, in Lemma 3.6, we show that our definition and the one given in [1] are equivalent. We also prove that the notion of integrally closed is a local property (Theorem 3.7). We apply the notion of integral extension of a module, and prove the Lying over, Going up and Going down theorems for modules (Theorems 3.10, 3.11, 3.12).

2. Integral elements of a module

Let M' be an R-module, and S be the set of regular elements of R. Then

$$T_{M'} = \{t \in S : tm' = 0, \text{ for some } m' \in M' \text{ implies that } m' = 0\}$$

is a multiplicative closed subset of R.

Definition 2.1. Let $M \subseteq M'$ be R-modules, $y \in T_{M'}^{-1}R$, and $m' \in M'$. We say that an element ym', is integral over M, if there exist a monic polynomial $f(x) \in R[x]$, and a polynomial $g(x) \in M[x]$ such that $deg(g(x)) < deg(f(x))$, and

$$f(y)m' + g(y) = 0.$$

Example 2.2. Let $M = 2\mathbb{Z}$, and $M' = \{a/2^n : a \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ be $\mathbb{Z}$-modules. Clearly, $T_{M'} = \mathbb{Z} - \{0\}$. Consider $m' = 1/2 \in M'$ and $y = 2/1 \in T_{M'}^{-1}\mathbb{Z}$. Then ym' is the integral over M, but $ym' \notin M$. If $m' = 1 \in M'$, and $y = 1/2 \in T_{M'}^{-1}\mathbb{Z}$, then ym' is not the integral over M.

Theorem 2.3. *Let $M \subseteq M'$ be R-modules. If $y \in T_{M'}^{-1}R$, and $m' \in M'$, then the following statements are equivalent:*
i) ym' is the integral over M;
ii) There exist a finitely generated R-module $L_1 = \Sigma_{i=1}^{n} Rx_i$, and a submodule K of an R-module $L_2 = \Sigma_{j=0}^{k} y^j M$, where $k < n$, such that $m' \in L_1$ and $yx_i \in L_1 + K$, for all $1 \leq i \leq n$;
iii) There exists an R-module L', such that $(L'/L) = \Sigma_{i=1}^{n} Rx'_i$, where $L = \Sigma_{j=0}^{k} y^j M$, and $k < n$ such that $m' \in L'$, and $yx'_i \in L'/L$ for all $1 \leq i \leq n$.

Proof. $(i) \Longrightarrow (ii)$ Since ym' is the integral over M, there exist

$$f(x) = x^n + \Sigma_{i=0}^{n-1} r_i x^i \in R[x]$$

and

$$g(x) = \Sigma_{i=0}^{k} m_i x^i \in M[x]$$

such that, $k < n$, and

$$f(y)m' + g(y) = y^n m' + \Sigma_{i=0}^{n-1} y^i r_i m' + \Sigma_{j=0}^{k} y^j m_j = 0.$$

Put

$$L_1 = Rm' + Rym' + \cdots + Ry^{n-1}m'$$

and

$$K = Rm_0 + Rym_1 + \cdots + Ry^k m_k.$$

Therefore, $y(y^{i-1}m') \in L_1$, for all $1 \leq i \leq n-1$. Since $f(y)m' + g(y) = 0$,

$$y(y^{n-1}m') = y^n m' = -(\Sigma_{i=0}^{n-1} r_i y^i m' + \Sigma_{j=0}^{k} y^j m_j) \in L_1 + K.$$

Hence,

$$y(y^{i-1}m') \in L_1 + K, \text{ for all } 1 \leq i \leq n.$$

$(ii) \Longrightarrow (iii)$ Put $L' = L_1 + L_2$. It is clear that L'/L_2 is generated by $x'_i = x_i + L_2$, and $yx'_i \in L'/L_2$, for all $1 \leq i \leq n$.

$(iii) \Longrightarrow (i)$ By assumption, there exist $r_{ij} \in R$, and $yx'_i = \Sigma_{j=1}^{n} r_{ij} x'_j$, for all $1 \leq i \leq n$. Hence,

$$\Sigma_{j=1}^{n} (\delta_{ij} y - r_{ij}) x'_j = 0.$$

Put $A = [\delta_{ij} y - r_{ij}]$ as an $n \times n$ matrix. Now, since $A^{adj}A = (detA)I$ where $A^{adj}, A,$ and I, are, respectively, the adjoint matrix of A, determinant of A, and identity matrix, We have

$$y^n m' + \cdots + r_1 y m' + r_0 m' + y^k m_k + \cdots + y m_1 + m_0 = 0$$

and so, ym' is the integral over M. □

Definition 2.4. Let $M \subseteq M'$ be R-modules, and $y \in T_{M'}^{-1}R$. We can define

$$\overline{M}^y_{M'} = \{ym' : m' \in M' \text{ and } ym' \text{ is the integral over } M\}.$$

Lemma 2.5. *Let $M \subseteq M'$ be R-modules, $y \in T_{M'}^{-1}R$, and $m'_1, m'_2 \in M'$. If ym'_1 and ym'_2 are the integrals over M, then $y(rm'_1 + sm'_2) = rym'_1 + sym'_2$ for all $r, s \in R$ is integral over M. Therefore, $\overline{M}^y_{M'}$ is an R-module.*

Proof. By Theorem 2.3, there exist R-modules $L_1 = \Sigma_{i=1}^{n_1} Rx_i$ and $L_2 = \Sigma_{i=1}^{n_2} Rz_i$, and submodules K_1 of $\Sigma_{i=0}^{k_1} y^i M$ and K_2 of $\Sigma_{i=0}^{k_2} y^i M$ such that $m'_1 \in L_1$, $m'_2 \in L_2$, $yx_i \in L_1 + K_1$, $yz_j \in L_2 + K_2$, $(1 \leq i \leq n_1, 1 \leq j \leq n_2)$, $k_1 < n_1$, and $k_2 < n_2$.

Put $L' = L_1 + L_2 + L$, where $L = \Sigma_{i=0}^{k} y^i M$ and $k = n_1 + n_2 - 1$. Therefore, L'/L is generated by the set

$$\{x_1 + L, \ldots, x_{n_1} + L, z_1 + L, \ldots, z_{n_2} + L\},$$

and $rm'_1 + sm'_2 \in L'$. However,

$$y(x_i + L) \in L'/L, \text{ and } y(z_j + L) \in L'/L \ (1 \leq i \leq n_1, 1 \leq j \leq n_2),$$

Thus by Theorem 2.3, $y(rm'_1 + sm'_2)$ is the integral over M. □

Corollary 2.6. *Let $N \subseteq M \subseteq M'$ be R-modules, and $T = T_{M'}$. If $y \in T^{-1}R$, then $\overline{N}^y_{M'} \subseteq \overline{M}^y_{M'}$ and $\overline{N}^y_{M} \subseteq \overline{N}^y_{M'}$.*

Corollary 2.7. *Let $N \subseteq M$ be R-modules, and $T = T_M$. If $y \in T^{-1}R$, then $\overline{N}^{y^{2k}}_{M} \subseteq \overline{N}^{y^k}_{M}$ for all $k \in \mathbb{N}$.*

Corollary 2.8. *Let $N \subseteq M$ be R-modules, and $T = T_M$. If $y \in T^{-1}R$, and $y' = ay$, for some $a \in R$, then $\overline{N}^{y}_{M} \subseteq \overline{N}^{y'}_{M}$.*

Theorem 2.9. *Let $M_1 \subseteq M'_1$ and $M_2 \subseteq M'_2$ be R-modules, and $T = T_{M'_1} \cap T_{M'_2}$. If $y \in T^{-1}R$, then $\overline{(M_1 \oplus M_2)}^y_{M'_1 \oplus M'_2} = \overline{(M_1)}^y_{M'_1} \oplus \overline{(M_2)}^y_{M'_2}$.*

Proof. Let $ym'_1 \in \overline{(M_1)}^y_{M'_1}$ and $ym'_2 \in \overline{(M_2)}^y_{M'_2}$. Since ym'_1 is the integral over M_1, it follows, from Theorem 2.3, that there exists a finitely generated R-module,

$$L_1 = \Sigma_{i=1}^{n_1} Rs_i,$$

such that, $m'_1 \in L_1$, and $ys_j \in L_1 + \Sigma_{i=0}^{k_1} y^i M_1$ for all $1 \leq j \leq n_1$, where $k_1 < n_1$. Similarly, since ym'_2 is the integral over M_2, there exists a finitely generated R-module,

$$L_2 = \Sigma_{i=1}^{n_2} Rt_i,$$

such that $m_2' \in L_2$, and $yt_j \in L_2 + \Sigma_{i=0}^{k_2} y^i M_2$, for all $1 \le j \le n_2$, where $k_2 < n_2$. Put

$$n = n_1 + n_2,\ k = n - 1,\ L_1' = L_1 + \Sigma_{i=0}^{k} y^i M_1,\ L_2' = L_2 + \Sigma_{i=0}^{k} y^i M_2,$$

and

$$T = \Sigma_{i=0}^{k} y^i (M_1 \oplus M_2).$$

Suppose that the R-module $(L_1' \oplus L_2')/T$ is generated by the set $A_1 \cup A_2$, where $A_1 = \{x_i : x_i = (s_i, 0) + T, 1 \le i \le n_1\}$, and $A_2 = \{x_i : x_i = (0, t_{i-n_1}) + T, n_1 + 1 \le i \le n_1 + n_2\}$. Clearly, $(m_1', m_2') \in L_1' \oplus L_2'$ and $yx_i = (ys_i, 0) + T \in (L_1' \oplus L_2')/T$ and $yx_j = (0, yt_{j-n_1}) + T \in (L_1' \oplus L_2')/T$, for all $1 \le i \le n_1$, and all $n_1 + 1 \le j \le n_1 + n_2$. Therefore, $y(m_1', m_2')$ is the integral over $M_1 \oplus M_2$.

Now, assume that $y(m_1', m_2')$ is the integral over $M_1 \oplus M_2$. We must show that ym_1' and ym_2' are the integrals over M_1 and M_2, respectively. By definition, there exist a monic polynomial $f(x) \in R[x]$, and $g(x) \in (M_1 \oplus M_2)[x]$, such that $deg(g(x)) < deg(f(x))$, and

$$f(y)(m_1', m_2') + g(y) = (y^n + r_{n-1} y^{n-1} + \cdots + r_1 y + r_0)(m_1', m_2') +$$
$$y^k (m_{1k}, m_{2k}) + \cdots + y(m_{11}, m_{21}) + (m_{10}, m_{20}) = 0.$$

Hence,

$$y^n m_1' + r_{n-1} y^{n-1} m_1' + \cdots + r_0 m_1' + y^k m_{1k} + \cdots + y m_{11} + m_{10} = 0,$$

and

$$y^n m_2' + r_{n-1} y^{n-1} m_2' + \cdots + r_0 m_2' + y^k m_{2k} + \cdots + y m_{21} + m_{20} = 0.$$

We can conclude that ym_1' is the integral over M_1, and ym_2' is the integral over M_2. □

Theorem 2.10. *Let $M \subseteq M'$ be R-modules, $y \in T_{M'}^{-1} R$, and S_1 be a multiplicative closed subset of R. Then $\overline{(S_1{}^{-1} M)}_{S_1{}^{-1} M'}^{y} = S_1{}^{-1}(\overline{M}_{M'}^{y})$.*

Proof. Let $s \in S_1$, and $ym' \in \overline{M}_{M'}^{y}$. Since ym' is the integral over M, there exist the positive integers $k < n$, $r_i \in R$, and $m_j \in M$ $(0 \le i \le n-1, 0 \le j \le k)$, such that

$$y^n m' + r_{n-1} y^{n-1} m' + \cdots + r_0 m' + y^k m_k + \cdots + y m_1 + m_0 = 0.$$

Hence, for $s \in S_1$, we have

$$y^n (m'/s) + r_{n-1} y^{n-1} (m'/s) + \cdots + r_0 (m'/s) +$$
$$y^k (m_k/s) + \cdots + y(m_1/s) + (m_0/s) = 0.$$

We may conclude that $y(m'/s)$ is the integral over $S_1^{-1} M$, and, therefore,

$S_1{}^{-1}(\overline{M}_{M'}^{y}) \subseteq \overline{(S_1{}^{-1} M)}_{S_1{}^{-1} M'}^{y}$.

Now, suppose that $m'/s \in S_1^{-1}M'$ and $y(m'/s)$ is the integral over ${S_1}^{-1}M$. Hence, there exist the positive integers $k < n$, $r_i \in R$, and $m_j/s_j \in S^{-1}M$ $(0 \leq i \leq n-1, 0 \leq j \leq k)$, such that

$$y^n(m'/s) + r_{n-1}y^{n-1}(m'/s) + \cdots + r_0(m'/s) + \\ y^k(m_k/s_k) + \cdots + y(m_1/s_1) + (m_0/s_0) = 0.$$

We have

$$y^n(s_0s_1 \ldots s_k)m' + \cdots + yr_1(s_0s_1 \ldots s_k)m' + r_0(s_0s_1 \ldots s_k)m' + \\ y^k(ss_0s_1 \ldots s_{k-1})m_{k-1} + \cdots + y(ss_0s_2 \ldots s_k)m_1 + (ss_1 \ldots s_k)m_0 = 0.$$

It follows that $y(s_0s_1 \ldots s_k)m'$ is the integral over M and so, $y(m'/s) \in {S_1}^{-1}(\overline{M}^y_{M'})$. Therefore, $\overline{({S_1}^{-1}M)}^y_{{S_1}^{-1}M'} \subseteq {S_1}^{-1}(\overline{M}^y_{M'})$. □

Theorem 2.11. *Let $M \subseteq M'$ be R-modules, and $y \in T^{-1}_{M'}R$. Then*

$$\overline{(M[x])}^y_{M'[x]} = (\overline{M}^y_{M'})[x].$$

Proof. Let $f(x) = \Sigma_{i=0}^n m'_i x^i \in M'[x]$, and $yf(x)$ be the integral over $M[x]$. There exist $h(z) \in R[z]$, and $g(z) \in (M[x])[z]$, such that $h(z)$ is monic, $deg(g(z)) < deg(h(z))$, and

$$h(y)f(x) + g(y) = \\ y^k f(x) + r_{k-1}y^{k-1}f(x) + \cdots + r_0 f(x) + y^\ell f_l(x) + \cdots + f_0(x) = 0.$$

We have

$$(y^k m'_n + \cdots + r_1 y m'_n + r_0 m'_n + y^\ell m_{\ell n} + \cdots + y m_{1n} + m_{0n})x^n + \\ \cdots + (y^k m'_0 + \cdots + r_0 m'_0 + y^\ell m_{\ell 0} + \cdots + m_{00}) = 0$$

and so,

$$y^k m'_n + \cdots + r_0 m'_n + y^\ell m_{\ell n} + \cdots + m_{0n} = 0.$$

Therefore, ym'_n is the integral over M. Similarly, $ym'_{n-1}, \ldots, ym'_0$ is the integral over M, and hence, $yf(x) \in (\overline{M}^y_{M'})[x]$.
Now, suppose that

$$f(x) = ym'_n x^n + \cdots + ym'_1 x + ym'_0 \in (\overline{M}^y_{M'})[x].$$

Thus ym'_i $(0 \leq i \leq n)$ are the integrals over M, and, by Theorem 2.3, there exist R-modules L_i $(0 \leq i \leq n)$, and positive integers $k_i < t_i$, such that L_i is generated by $\{a_{1i}, \ldots, a_{t_i i}\}$, and $ya_{ji} \in L_i + \Sigma_{t=0}^{k_i} y^t M$, $0 \leq i \leq n$, and $1 \leq j \leq t_i$. Put

$$t = \Sigma_{i=0}^n t_i, \;\; k = t - 1 \text{ and } L = L_0 + L_1 x + \ldots L_n x^n.$$

Then,

$$L = Ra_{10} + \cdots + Ra_{t_0 0} + \\ R(a_{11}x) + \cdots + R(a_{t_1 1}x) + \cdots + R(a_{1n}x^n) + \cdots + R(a_{t_n n}x^n),$$

and $f(x) \in L$. Furthermore

$$ya_{ij}x^j \in L + \Sigma_{t=0}^{k} y^t M[x],\ 1 \leq i \leq n,\ 1 \leq j \leq t_i$$

and, hence, by Theorem 2.3, $yf(x)$ is the integral over $M[x]$. □

3. Integral extension of a module

Definition 3.1. Let $M \subseteq M'$ be torsion-free R-modules, and $T_{M'} = R - \{0\}$. We say that M' is an integral extension of M, if ym' is the integral over M, for all $y \in T_{M'}^{-1}R$, and $m' \in M'$.

Example 3.2. Let $V \subseteq V'$ be the vector spaces over a field F. Thus $T_{V'} = F - \{0\}$, and $T_{V'}^{-1}F = F$. Suppose that $y \in T_{V'}^{-1}F$, and $v' \in V'$. We have $f(x) = x - y \in F[x]$, and $g(x) = 0 \in V[x]$. Hence, $f(y)v' + g(y) = 0$. Therefore, V' is an integral extension of V.

Proposition 3.3. *Let $M \subseteq M' \subseteq M''$ be the torsion-free R-modules, and $T = R - \{0\}$. If M' is an integral extension of M, and M'' is an integral extension of M',. then M'' is an integral extension of M.*

Proof. Let $y \in T^{-1}R$, and $m'' \in M''$. Since ym'' is the integral over M', there exist the positive integer $k < n$, $r_i \in R$, and $m'_j \in M'$, such that

$$y^n m'' + r_{n-1}y^{n-1}m'' + \cdots + r_1 ym'' + r_0 m'' + y^k m'_k + \cdots + ym'_1 + m'_0 = 0.$$

But ym'_i , $1 \leq i \leq k$, are the integrals over M, and hence, by Theorem 2.3, there exist the positive integers $k_i < n_i$, $m_{0i}, \ldots, m_{k_i i} \in M$, and $s_{0i}, s_{1i}, \ldots, s_{n_{i-1}i} \in R$, such that

$$y^{n_i}m'_i + y^{n_i-1}s_{in_i-1}m'_i + \cdots + s_{i0}m'_i + y^{k_i}m_{ik_i} + \cdots + ym_{i1} + m_{i0} = 0.$$

Define

$$n'_i = \begin{cases} n_i & if \quad n_i \geq i \\ i & if \quad n_i < i \end{cases}$$

Now, put

$$L = Rm'' + R(ym'') + \cdots + R(y^{n-1}m'') + L_1 + \cdots + L_k;$$

$$t = n + n'_1 + \cdots + n'_k - 1,$$

where $L_i = Rm'_i + R(ym'_i) + \cdots + R(y^{n'_i-1}m'_i),\ \ 1 \leq i \leq k,$

$K = \Sigma_{i=0}^{t} y^i M$. Then, by Theorem 2.3, ym'' is the integral over M. □

Note: Let $M \subseteq M'$ be R-modules, and $y \in T_{M'}^{-1}R$, $m' \in M'$. If ym' is integral over $\overline{M}^y$, then $ym' \in \overline{M}^y$.

Let M be an R-module. In [1], Alkan and Tiras have defined the integrally closed for M, as follows: if for any $y \in T_M^{-1}R$, $m \in M$, such that $y^n m + y^{n-1}m_{n-1} + \cdots + ym_1 + m_0 = 0$; $m_i \in M$ $(0 \leq i \leq n)$, then $ym \in M$. In what follows, we define the notion of integrally closed modules with the integral elements, and show that the two definitions are equivalent.

Definition 3.4. Let M be an R-module. We say that M is integrally closed, if $M = \Sigma_{y \in T_M^{-1}R} \overline{M}^y$, where

$$\overline{M}^y = \{ym : m \in M \text{ and } ym \text{ is integral over } M\}.$$

Example 3.5. Let $R = \mathbb{Z}$, p be a prime integer, and $L = \{a/b : a, b \in \mathbb{Z}, p \nmid b\}$. Consider $M = L/\mathbb{Z}$. Then $T = \{p^n : n \in \mathbb{N} \cup \{0\}\}$ and so, $T^{-1}R = \{z/p^n : z \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$. Suppose that $z/p^k \in T^{-1}R$, $(z,p) = 1$, and $a/b \in M$, such that $(z/p^k)(a/b)$ is the integral over M. Thus there exist a monic polynomial $f(x) \in R[x]$, and a polynomial $g(x) \in M[x]$, such that $l = deg(g(x)) < deg(f(x)) = n$, and

$$0 = f(y)m' + g(y) =$$

$$(z/p^k)^n a/b + (z/p^k)^{n-1} r_{n-1}(a/b) + ... + r_0(a/b) + (z/p^k)^l (a_l/b_l) + ... + (a_0/b_0),$$

where $r_{n-1}, ..., r_0 \in R$, $a_i/b_i \in M$, for all i, $0 \leq i \leq l$. We can conclude $z^n a(b_l ... b_0) = p^k c$ for some integer c. Since $(p,z) = 1$, and $(p, b_i) = 1$, for all i, $1 \leq i \leq l$, $p^k \mid a$. Thus $(z/p^k)(a/b) \in M$. Therefore, M is an integrally closed module.

Lemma 3.6. *Let M be an R-module. Then M is integrally closed if and only if for any $y \in T^{-1}R$, and $m \in M$ such that $y^n m + y^{n-1}m_{n-1} + \cdots + ym_1 + m_0 = 0$, with $m_i \in M$ $(0 \leq i \leq n-1)$, implies that $ym \in M$.*

Proof. Let M be integrally closed, $y \in T^{-1}R$, and $m \in M$, such that,

$$y^n m + y^{n-1}m_n + \cdots + ym_1 + m_0 = 0, m_i \in M (0 \leq i \leq n).$$

Since M is integrally closed, $\overline{M}^y \subseteq M$. But $ym \in \overline{M}^y$, and hence, $ym \in M$. Conversely, suppose that $ym \in \overline{M}^y$. Then there exist $f(x) \in R[x]$, and $g(x) \in M[x]$, such that $f(x)$ is monic, $deg(g(x)) < deg(f(x))$, and

$$f(y)m + g(y) =$$

$$y^n m + r_{n-1}y^{n-1}m + \cdots + r_0 m + y^k m_k + \cdots + ym_1 + m_0 = 0.$$

By assumption $ym \in M$, and so $\overline{M}^y \subseteq M$. Therefore, $\Sigma_{y \in T^{-1}R} \overline{M}^y \subseteq M$. Since $\overline{M}^1 = M$, it follows that $M = \Sigma_{y \in T^{-1}R} \overline{M}^y$. We can conclude that M is integrally closed. $\square$

Proposition 3.7. *Let M be a torsion free R-module. Then the following statements are equivalent:*
i) M is integrally closed;

ii) M_P *is integrally closed, for each prime ideal* P *of* R*;*
iii) M_Q *is integrally closed, for each maximal ideal* Q *of* R*.*

Proof. $(i) \Longrightarrow (ii)$ Let M be integrally closed and P be a prime ideal of R. Thus $T_{M_P} = R_P - \{0\}$. Suppose that $(a/t)/(b/t') \in T_{M_P}^{-1}R_P$, $m/s \in M_P$, and $((a/t)/(b/t'))(m/s)$ is the integral over M_P. There exists a monic polynomial $f(x) \in R_P[x]$, a polynomial $g(x) \in M_P[x]$, and $l = deg(g(x)) < deg(f(x)) = k$, such that

$$f((a/t)/(b/t'))(m/s) + g((a/t)/(b/t')) =$$

$$((a/t)/(b/t'))^k(m/s) + ... + ((a/t)/(b/t'))(r_1/s_1)(m/s) + (r_0/s_0)(m/s) +$$
$$((a/t)/(b/t'))^l(m_l/s'_l) + ... + (m_0/s'_0) = 0,$$

where $m_i/s'_i \in M_P$, for all i, $0 \leq i \leq l$, and $r_j/s_j \in R_P$, for all j, $0 \leq j \leq k-1$. We can conclude

$$(at'/bt)^k(s''m) + (at'/bt)^{k-1}(m'_{k-1}) + ... + (at'/bt)m'_1 + m'_0 = 0,$$

where $s'' = s_{k-1}...s_1s_0s'_l...s'_1s'_0$, and $m'_{k-1}, ..., m'_0 \in M$.
Hence, $(at'/bt)(s''m) \in M$. Thus there exists $m' \in M$, such that $(a/t)(m/s) = (b/t')(m'/ss'')$. Therefore, $((a/t)/(b/t'))(m/s) \in M_P$.
$(ii) \Longrightarrow (iii)$ It is clear.
$(iii) \Longrightarrow (i)$ Since M is a torsion free R-module, $T = R - \{0\}$. Suppose that $a/b \in T^{-1}R$, $m \in M$, and $(a/b)m$ is the integral over M. Thus there exists a monic polynomial $f(x) \in R[x]$, a polynomial $g(x) \in M[x]$, and $l = deg(g(x)) < deg(f(x)) = k$, such that

$$f(a/b)m + g(a/b) =$$

$$(a/b)^k m + ... + (a/b)r_1 m + r_0 m + (a/b)^l m_l + ... + (a/b)m_1 + m_0 = 0,$$

where $r_{k-1}, ..., r_1, r_0 \in R$, and $m_l, ..., m_1, m_0 \in M$. Consider the subset $I = \{r \in R : ram = bm', \text{ for some } m' \in M\}$. It is clear that I is an ideal of R. Assume that I is a proper ideal of R. There exists a maximal ideal Q for R such that, $I \subseteq Q$. We have

$$[(a/1)/(b/1)]^k(m/1) + ... + [(a/1)/(b/1)](r_1/1)(m/1) + (r_0/1)(m/1) +$$
$$[(a/1)/(b/1)]^l(m_l/1) + ... + [(a/1)/(b/1)](m_1/1) + (m_0/1) = 0.$$

Hence $((a/1)/(b/1))(m/1)$ is integral over M_Q. Since M_Q is integrally closed, there exists $m'/s \in M_Q$ such that $((a/1)/(b/1))(m/1) = m'/s$. Then $sam = bm'$, and so, $s \in I \subseteq Q$, which is a contradiction. Hence, $I = R$ so $(a/b)m \in M$. □

Now, we apply the notion of integral extension of a module, and prove the Lying over, Going up and Going down theorems for modules. We need the following two lemmas.

Lemma 3.8. *Let $M \subseteq M'$ be the torsion-free R-modules, and $P \in Spec(M)$. If M' is the integral over M, then $(P : M) = 0$.*

Proof. Let $0 \neq r \in (P : M)$, and $m \in M \backslash P$. Since M' is the integral over M, $(1/r)m$ is the integral over M. Hence there exist $f(x) \in R[x]$, and $g(x) \in M[x]$, such that $f(x)$ is monic, $deg(g(x)) < deg(f(x))$, and

$$f(1/r)m + g(1/r) =$$

$$(1/r)^n m + (1/r)^{n-1} r_{n-1} m + \cdots + r_0 m + (1/r)^k m_k + \cdots + (1/r)m_1 + m_0 = 0.$$

Therefore,

$$m = -r(r_{n-1}m + \cdots + r_1 r^{n-2} m + r_0 r^{n-1} m + r^{n-1-k} m_k + \cdots + r^{n-1} m_1 + r^n m_0),$$

and since $r \in (P : M)$, it follows that $m \in P$, which is a contradiction. Thus $(P : M) = 0$. □

Lemma 3.9. *Let $M \subseteq M'$ be the R-modules, and $Q \in Spec(M')$. If $Q \cap M = P \neq M$, then $P \in Spec(M)$ and $(Q : M') = (P : M)$.*

Proof. Let $r \in (P : M)$, and $m \in M \backslash P$. Since $rm \in P \subseteq Q$ and $m \notin Q$, $r \in (Q : M')$, hence $(P : M) \subseteq (Q : M')$. Now, suppose that $r \in (Q : M')$ and $m \in M$. Then $rm \in Q \cap M = P$ and so $r \in (P : M)$. Hence $(P : M) = (Q : M')$. Now we show that $P \in Spec(M)$. Let $m \in M$, $r \in R$, and $rm \in P$. Since $rm \in Q$, and $Q \in Spec(M')$, it follows that $m \in Q$ or $r \in (Q : M')$. Therefore, $m \in P$ or $r \in (P : M)$ and so, $P \in Spec(M)$. □

Theorem 3.10. *(Lying over). Let $M \subseteq M'$ be the torsion-free R-modules and M' be integral over M. If $P \in Spec(M)$, then there exists $Q \in Spec(M')$, such that $Q \cap M = P$.*

Proof. Let $P \in Spec(M)$. Put $\mathfrak{A} = \{P' \leq M | P' \cap M = P\}$. Since $P \in \mathfrak{A}$, $\mathfrak{A} \neq \emptyset$, by Zorn's Lemma, $\mathfrak{A}$ has a maximal element Q. Now, we show that $Q \in Spec(M')$. Since $Q \cap M = P \neq M$, we have $Q \neq M'$. Suppose that $r \in R$, $m' \in M'$, such that $rm' \in Q$, and $m' \notin Q$. Since $(Q + Rm') \cap (M \backslash P) \neq \emptyset$, there exist $t \in R$, and $q \in Q$, such that $q + tm' = m \notin P$. Hence, $rq + rtm' = rm \in P$. But $P \in Spec(M)$, and therefore, $r \in (P : M)$. By Lemma 3.8, $r = 0 \in (Q : M')$, and we can conclude that $Q \in Spec(M')$. □

Theorem 3.11. *(Going up). Let $M \subseteq M'$ be the torsion-free R-modules, and M' be the integral over M. If $P_0 \subseteq P_1$ are the prime submodules of M, and $Q_0 \in Spec(M')$, such that $Q_0 \cap M = P_0$, then there exists $Q_1 \in Spec(M')$, such that $Q_0 \subseteq Q_1$, and $Q_1 \cap M = P_1$.*

Proof. Put $\mathfrak{A} = \{L \leq M' | Q_0 \subseteq L, L \cap M = P_1\}$. Since $Q_0 \cap M = P_0$, it follows that $(Q_0 + P_1) \in \mathfrak{A}$. Hence $\mathfrak{A} \neq \emptyset$. By Zorn's Lemma, $\mathfrak{A}$ has a maximal element Q_1. We show that $Q_1 \in Spec(M')$. Since $Q_1 \cap M = P_1 \neq M$, it follows that $Q_1 \neq M'$. Let $m' \in M'$, $r \in R$ such that $rm' \in Q_1$. Suppose that $m' \notin Q_1$. Since Q_1 is a maximal element of $\mathfrak{A}$, $Q_1 + \langle m' \rangle \notin \mathfrak{A}$. Since $Q_0 \subseteq Q_1 + \langle m' \rangle$, $P_1 \subset Q_1 + \langle m' \rangle \cap M$. Hence, there exist $q \in Q_1$, and $t \in R$, such that $q + tm' \in Q_1 \cap M$, and $q + tm' \notin P_1$. We have $rq_1 + rtm' \in Q_1 \cap M = P_1$. Since $P_1 \in Spec(M)$, it follows that $r \in (P_1 : M)$. By Lemma 3.8, $r = 0 \in (Q_1 : M')$, we can conclude that $Q_1 \in Spec(M')$. □

Theorem 3.12. *(Going down). Let $M \subseteq M'$ be the torsion-free R-modules, and M' be the integral over M. If $P_0 \subseteq P_1$ are prime submodules of M, and $Q_1 \in Spec(M')$, such that $P_1 = Q_1 \cap M$. Then there exists $Q_0 \in Spec(M')$, such that $Q_0 \subseteq Q_1$, and $Q_0 \cap M = P$.*

Proof. Put $\mathfrak{A} = \{L \leq M' | L \subseteq Q_1, L \cap M = P_0\}$. Since $P_0 \in \mathfrak{A}$, $\mathfrak{A} \neq \emptyset$. By Zorn's Lemma, $\mathfrak{A}$ has a maximal element Q_0. We show that $Q_0 \in Spec(M')$. Suppose that $r \in R$, $m' \in M'$, such that $rm' \in Q_0$. If $Q_0 + Rm' \subseteq Q_1$, then $(Q_0 + Rm') \cap (M \backslash P_0) \neq \emptyset$, and so, there exist $t \in R$, and $q \in Q_0$, such that $m = q + tm' \notin P_0$. Since $rm = rq + rtm' \in P_0 \in Spec(M)$, $r \in (P_0 : M) = 0$. Therefore, $r \in (Q_0 : M')$. But, if $Q_0 + Rm' \nsubseteq Q_1$, then there exist $t \in R$, and $q_0 \in Q_0$, such that $q_0 + tm' \notin Q_1$. Since $rq_0 + rtm' \in Q_1$, and $Q_1 \in Spec(M')$, it follows that, $r \in (Q_1 : M')$. Hence, by Lemmas 3.8, and 3.9, $(Q_1 : M') = (P_1 : M) = 0$, and this implies that $r \in (Q_0 : M')$. □

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AN INTEGRAL DEPENDENCE IN MODULES OVER COMMUTATIVE RINGS

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وابستگی صحیح در مدول‌های روی حلقه‌های جابه‌جایی

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در این مقاله، مفهوم وابسته صحیح در حلقه‌ها را به مدول‌ها گسترش می‌دهیم. پایایی بستار صحیح نسبت به ساختارهای متفاوت مدول را مطالعه می‌کنیم. در ادامه، توسیع صحیح مدول را معرفی کرده و قضایای رو قرار داشتن، بالارفتن و پایین رفتن را برای مدول‌ها ثابت می‌نماییم.

کلمات کلیدی: زیرمدول اول، عنصر صحیح، به‌طور صحیح وابسته.