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GENERALIZED PRINCIPAL IDEAL THEOREM FOR MODULES

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Abstract. The generalized principal Ideal theorem is one of the cornerstones of the dimension theory for the Noetherian rings. For an R-module M, certain submodules of M that play a role analogous to that of prime ideals in the ring R are identified. Using this definition, we extend the this theorem to modules.

1. Introduction

Throughout this paper, all rings are commutative with identity, and all modules are unitary. Also we consider R to be a ring, and M to be an R-module.

For a submodule N of M, let $(N :_R M)$ denote the set of all elements r in R, such that $rM \subseteq N$. Note that $(N :_R M)$ is an ideal of R. The annihilator of M, denoted by $\mathrm{Ann}_R(M)$, is $(0 :_R M)$. When there is no ambiguity, we just write $(N : M)$ instead of $(N :_R M)$. A proper submodule P of M is said to be *prime,* if $rx \in P$, for $r \in R$, and $x \in M$, implies that either $x \in P$ or $r \in (P : M)$. The concept of prime submodule has been introduced by Dauns [5], and it has recently been studied extensively by various authors (see, for example, [13], [14], [15], and [16]). The collection of all prime submodules of M is denoted by $\mathrm{Spec}_R(M)$, and the collection of all maximal submodules of M is denoted by $\mathrm{Max}_R(M)$.

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In this article, we introduce a slightly different notion of the prime submodule.

Definition. Let P be a proper submodule of an R-module M. We say that P is a *completely prime* submodule of M, if there is a prime ideal $\mathfrak{p}$ of R, such that $M/P \cong R/\mathfrak{p}$, as R-modules.

The set of all completely prime submodules of an R-module M is denoted by C-Spec$_R(M)$.

Note that if we consider R as an R-module, then the completely prime submodules are exactly the prime ideals of R.

The generalized principal ideal Theorem asserts that if R is a Noetherian ring, and $\mathfrak{p}$ is a minimal prime of an ideal $(a_1, \ldots, a_n)$ of R, then $\mathrm{ht}(\mathfrak{p}) \leq n$. This theorem is one of the cornerstones of the dimension theory for Noetherian rings; see, for example, [6, Theorem 10.1]. Indeed, [12, p. 104] have call it "the most important single theorem in the theory of Noetherian rings". This theorem provides a lower estimate of the number of generators of an ideal in a Noetherian ring and the number of equations needed to describe an algebraic variety. This theorem has been studied intensively in the literature; see, for example, [1], [9], [11], and [17].

Nishitani, in [17] has extended the generalized principal ideal theorem to modules. The aim of this paper is to give an alternative extension of this theorem to modules.

2. Completely Prime Submodules

We begin with the following proposition.

Proposition 2.1. *Let M be an R-module, and P be a submodule of M. Then the followings are equivalent.*

1. *P is a completely prime submodule of M.*
2. *P is a prime submodule of M, and $M/P \cong R/(P : M)$ as R-modules.*
3. *P is a prime submodule of M, and M/P is a cyclic R-module.*

Proof. (1) $\Longrightarrow$ (2): Let P be a completely prime submodule of M. Thus there is a prime ideal $\mathfrak{p}$ of R, such that $M/P \cong R/\mathfrak{p}$, as R-modules. It follows easily from [15, Proposition 1.2] that P is a prime submodule of M. On the other hand, $\mathfrak{p} = \mathrm{Ann}_R(R/\mathfrak{p}) = \mathrm{Ann}_R(M/P) = (P : M)$.

(2) $\Longrightarrow$ (3): Trivial.

(3) $\Longrightarrow$ (1): Let P be a prime submodule of M, and M/P be a cyclic R-module. Thus $(P : M)$ is a prime ideal, and there exists an ideal I of R, such that $M/P \cong R/I$. It follows that $I = (P : M)$, and hence, P is a completely prime submodule of M. □

Corollary 2.2. *Let M be an R-module. Then the followings hold.*

(1) $\text{C-Spec}_R(M) \subseteq \text{Spec}_R(M)$.
(2) $\text{Max}_R(M) \subseteq \text{C-Spec}_R(M)$ *and equality hold if R is a field.*

Proof. (1): Follows easily from the above proposition.
(2): Let $P \in \text{Max}_R(M)$. Then P is a prime submodule of M, and M/P is a simple R-module. Therefore, $P \in \text{C-Spec}_R(M)$, by the above proposition.

Now, let R be a field, and $P \in \text{C-Spec}_R(M)$. Then $M/P \cong R/(P : M) = R$. It follows that $P \in \text{Max}_R(M)$. □

The following example shows that a prime submodule need not be a completely prime submodule.

Example 2.3. Let R be a ring, and $\mathfrak{p} \in \text{Spec}(R)$. Then $\mathfrak{p} \times \mathfrak{p}$ is a prime submodule of the R-module $R \times R$. But we claim that $\mathfrak{p} \times \mathfrak{p}$ is not a completely prime submodule. Suppose to the contrary, that $\mathfrak{p} \times \mathfrak{p}$ is a completely prime submodule. Then Proposition 2.1 implies that

$$\frac{R}{\mathfrak{p}} \times \frac{R}{\mathfrak{p}} \cong \frac{R \times R}{\mathfrak{p} \times \mathfrak{p}} \cong \frac{R}{(\mathfrak{p} \times \mathfrak{p}) :_R (R \times R)} = \frac{R}{\mathfrak{p}},$$

contradicting the fact that $\frac{R}{\mathfrak{p}} \times \frac{R}{\mathfrak{p}}$ is not an integral domain.

Theorem 2.4. *Let M be an R-module. Then* $\text{C-Spec}_R(M) \neq \emptyset$, *if and only if* $\text{Max}_R(M) \neq \emptyset$.

Proof. If $\text{Max}_R(M) \neq \emptyset$, Corollary 2.2(2) implies that $\text{C-Spec}_R(M) \neq \emptyset$. Now, suppose that $\text{C-Spec}_R(M) \neq \emptyset$. Let $P \in \text{C-Spec}_R(M)$. Then the module M/P is cyclic, and has a maximal submodule. Therefore, M has a maximal submodule, i.e. $\text{Max}_R(M) \neq \emptyset$. □

A ring R is called *Max-ring*, if every R-module has a maximal submodule. Max-rings, which is also called *B-rings*, has been introduced by [10] and has been studied by several authors; see, for example, [4], [8], and [19].

The followings corollary provides characterizations of Max-rings.

Corollary 2.5. *Let R be a ring. Then the following are equivalent.*

(1) R is a Max-ring.
(2) Every R-module has a completely prime submodule.
(3) Every R-module has a prime submodule.

Proof. (1)$\Longrightarrow$(2) follows from Corollary 2.2(2).
(2)$\Longrightarrow$(3) follows from Corollary 2.2(1).
(3)$\Longrightarrow$(1) follows from [2, Theorem 3.9]. □

Proposition 2.6. *Let M be a finitely generated R-module, and U be a multiplicatively closed subset of R. Then*

$$\{U^{-1}P | P \in \text{C-Spec}_R(M) \ \textit{and}\ (P:M) \subseteq R\backslash U\} \subseteq \text{C-Spec}_{U^{-1}R}(U^{-1}M).$$

Proof. Let $P \in \text{C-Spec}_R(M)$, and let U be a multiplicatively closed subset of R, such that $(P:M) \subseteq R \setminus U$. It is easy to see that $U^{-1}P$ is a prime submodule of $U^{-1}M$. Now, we can show that $U^{-1}P \in \text{C-Spec}_{U^{-1}R}(U^{-1}M)$. Since $P \in \text{C-Spec}_R(M)$, we have $M/P \cong R/(P :_R M)$, as R-modules. From parts (i) and (ii) of [18, Lemma 9.12], we have $U^{-1}M/U^{-1}P \cong U^{-1}R/(U^{-1}P :_{U^{-1}R} U^{-1}M)$, as $U^{-1}R$-modules. Therefore, $U^{-1}P \in \text{C-Spec}_{U^{-1}R}(U^{-1}M)$. □

The following example shows that, in general, the containment in Proposition 2.6 can be strict.

Example 2.7. Let $\mathbb{Z}$ be the ring of integers $M = \mathbb{Z}\times\mathbb{Z}$ and $U = \mathbb{Z}\backslash\{0\}$. Then it is easy to see that

$$\text{C-Spec}_R(M) = \{\mathfrak{p} \times R | \mathfrak{p} \in \operatorname{Spec} R\} \cup \{R \times \mathfrak{p} | \mathfrak{p} \in \operatorname{Spec} R\}.$$

Therefore, $\{U^{-1}P | P \in \text{C-Spec}_R(M)$ and $(P:M) \subseteq R \setminus U\} = \{0 \times \mathbb{Q}, \mathbb{Q} \times 0\}$. On the other hand,

$$\text{C-Spec}_{U^{-1}R}(U^{-1}M) = \text{C-Spec}_{\mathbb{Q}}(\mathbb{Q} \times \mathbb{Q}) = \text{Max}_{\mathbb{Q}}(\mathbb{Q} \times \mathbb{Q}),$$

which has infinite elements.

We end this section by the following lemma which is used widely in the sequel.

Lemma 2.8. *Let M be an R-module, and let $P_1 \subsetneq P_2$ in* $\text{C-Spec}_R(M)$. *Then*

$$(P_1 : M) \subsetneq (P_2 : M).$$

Proof. Since $P_1 \subseteq P_2$, we have $(P_1 : M) \subseteq (P_2 : M)$. Suppose, to the contrary, that $(P_1 : M) = (P_2 : M) := \mathfrak{p}$. Now, let $T = R/\mathfrak{p}$. We have

$$T \cong M/P_1 \xrightarrow{\pi} M/P_2 \cong T,$$

where π is the natural surjective T-homomorphism. Clearly, every surjective T-homomorphism in $\operatorname{End}_T(T)$ is one to one. Therefore, P_1 must be equal to P_2, which is a contradiction. Hence $(P_1 : M) \subsetneq (P_2 : M)$. □

3. A Generalized Principal Ideal Theorem for Modules

For an ideal I of R, the height of I is denoted by $\mathrm{ht}(I)$. The generalized principal ideal theorem states that if R is a Noetherian rings, and $\mathfrak{p}$ is a minimal prime ideal of an ideal $(a_1, \ldots, a_n)$ generated by n elements of R, then $\mathrm{ht}(\mathfrak{p}) \leq n$. Consequently, $\mathrm{ht}(a_1, \ldots, a_n) \leq n$.

One might ask whether this theorem can be extended to modules. Nishitani in [17], has proved that it holds for modules. The aim of this section is to give an alternative generalization of this theorem to modules. For this purpose, we need to define some notions.

Let P be a completely prime submodule of M. We can say that P is a *completely minimal prime* over a submodule N of M, if $N \subseteq P$, and does not exist as a completely prime submodule K of M, such that $N \subseteq K \subsetneq P$.

Definition 3.1. (1) Let P be a completely prime submodule of M. The *completely height* of P, denoted by $\text{c-ht}_R(P)$, is defined by $\text{c-ht}_R(P) = \sup\{n | \exists\ P_0, P_1, \ldots, P_n \in \text{C-Spec}_R(M)$ such that $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P\}$.
(2) Let N be a proper submodule of an R-module M. The *completely height* of N, denoted $\text{c-ht}_R(N)$, is defined by $\text{c-ht}_R(N) = \min\{\text{c-ht}_R(P) | P \in \text{C-Spec}_R(M), P$ is a completely minimal prime over $N\}$.

Lemma 3.2. *Let M be a finitely generated R-module, $P \in \text{C-Spec}_R(M)$ and $U = R \setminus (P : M)$. Then $\text{c-ht}_R(P) \leq \text{c-ht}_{U^{-1}R}(U^{-1}P)$.*

Proof. Consider the following chain of distinct completely prime submodules of M

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t = P.$$

We can show that this chain induces the following chain of distinct completely prime submodules of $U^{-1}M$.

$$U^{-1}P_0 \subsetneq U^{-1}P_1 \subsetneq \cdots \subsetneq U^{-1}P_t = U^{-1}P.$$

Let $i \in \{0, 1, \ldots, t-1\}$. Proposition 2.6 implies that $U^{-1}P_i$ is a completely prime submodule of $U^{-1}M$. It is easy to see that $U^{-1}P_i \subseteq U^{-1}P_{i+1}$. Now, assume that $U^{-1}P_i = U^{-1}P_{i+1}$. Let $x \in P_{i+1} \setminus P_i$. Then there exist $y \in P_i$, and $u \in U$, such that $x/1 = y/u$. Therefore, $utx \in P_i$, for some $t \in U$. Since $tu \notin (P_i : M)$, we must have $x \in P_i$, which is a contradiction. Therefore, $U^{-1}P_i \subsetneq U^{-1}P_{i+1}$. This completes the proof. □

The following example shows that, in general, the inequality in Lemma 3.2 can be strict.

Example 3.3. Let K be a field, $n \geq 2$ be a natural number, and $R = K[x_1, x_2, \ldots, x_n]$ denotes the polynomial ring in a finite number of indeterminates $x_1, x_2, \ldots, x_n$. Let $A = Rx_1 + Rx_2 + \cdots + Rx_n$,

$B = R(x_1 - x_1^2) + R(x_2 - x_2^2) + \cdots + R(x_n - x_n^2)$, and $M = A/B$. By [7, Corollary 1.3], M is a multiplication ideal of the ring R/B, and hence, M is a multiplication R-module. Since M is not cyclic, $0 \notin \text{C-Spec}_R(M)$. Now, let $U = R \setminus \mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R. By [3, Proposition 5], $0 \in \text{C-Spec}_{U^{-1}R}(U^{-1}M)$. Consider the following chain of distinct completely prime submodules of M

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t = P.$$

As the proof of Lemma 3.2, this chain induces the following chain of distinct completely prime submodules of $U^{-1}M$.

$$0 \subsetneq U^{-1}P_0 \subsetneq U^{-1}P_1 \subsetneq \cdots \subsetneq U^{-1}P_t = U^{-1}P.$$

Hence, $\text{c-ht}_R(P) + 1 \leq \text{c-ht}_{U^{-1}R}(U^{-1}P)$.

Now, we are at a position to prove the main result of this paper.

Theorem 3.4. *Let R be a ring, and M be a Noetherian flat R-module. Let N be a proper submodule of M, generated by n elements $x_1, \ldots, x_n \in M$. Then* $\text{c-ht}_R N \leq n$.

Proof. Replacing $R/(0 : M)$ by R, we may suppose that R is a Noetherian ring. Let $\text{c-ht}_R N = t$. Then there is a completely minimal prime submodule P over N such $\text{c-ht}_R(P) = t$. Let $\mathfrak{p} = (P : M)$, and $U = R \setminus \mathfrak{p}$. In view of the above lemma, we have $\text{c-ht}_R(N) \leq \text{c-ht}_{U^{-1}R}(U^{-1}N)$. Thus, replacing $U^{-1}R$ by R, we may suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{p}$. Since M is a finitely generated flat module over a local ring, it is free of finite rank r. Let $\{e_1, e_2, \ldots, e_r\}$ be a base for M. Since $M/P \cong R/\mathfrak{p}$ as R-modules, we may suppose that $e_1, e_2, \ldots, e_{r-1} \in P$, $e_r \notin P$, and $P = Re_1 + Re_2 + \cdots + Re_r + \mathfrak{p}e_r$. There are elements $a_{1j}, a_{2j}, \ldots, a_{r-1j} \in R$, and $a_{rj} \in \mathfrak{p}$, such that $x_j = a_{1j}e_1 + a_{2j}e_2 + \cdots + a_{rj}e_r$. Let $\mathfrak{q}$ be a minimal prime ideal over an ideal $(a_{r1}, a_{r2}, \ldots, a_{rn})$ and Q denotes the submodule $Re_1 + Re_2 + \cdots + Re_r + \mathfrak{q}e_r$. Since $M/Q \cong R/\mathfrak{q}$, Q is a completely prime submodule of M. Therefore, $P = Q$, by the minimality of P. Hence, $\mathfrak{p} = \mathfrak{q}$, and so, $\mathfrak{p}$ is a minimal prime over an ideal, generating by n elements. Since $\text{c-ht}_R(P) = t$, we can consider the following chain of distinct completely prime submodules of M

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t = P.$$

By Lemma 2.8, the above chain induces a chain

$$(P_0 : M) \subsetneq (P_1 : M) \subsetneq \cdots \subsetneq (P_t : M) = \mathfrak{p}$$

of distinct prime ideals of R. Now, by the generalized principal ideal theorem for rings, we have $t \leq \mathrm{ht}_R(\mathfrak{p}) \leq n$. This completes the proof. □

Finally, we give an example to show that the above theorem is not correct without the Noetherian condition.

Example 3.5. Let $\mathbb{Z}$ be the ring of integers, $M = R = \mathbb{Z}[x_1, x_2, \ldots]$, $K =< 2 >= \mathbb{Z}[x_1, x_2, \ldots]$, and $\mathfrak{p}_i = 2\mathbb{Z}[x_{i+1}, x_{i+2}, \ldots]$, for all $i \geq 1$. Then $\mathfrak{p}_i$'s are prime ideals of R. Since

$$K \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \mathfrak{p}_3 \supsetneq \cdots,$$

we must have c-$\mathrm{ht}_R K = \infty$.

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قضیه ایده ال اصلی تعمیم یافته برای مدول ها

علیرضا نقی پور
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قضیه ایده ال اصلی تعمیم یافته یکی از سنگ بناهای نظریه بعد برای حلقه های نوتری است. برای $R-$ مدول M، زیرمدول های معینی را معرفی خواهیم کرد که رفتاری شبیه به ایده ال های اول حلقه R دارند. با به کار بردن این تعریف قضیه ایده ال اصلی تعمیم یافته را برای مدول ها گسترش خواهیم داد.

کلمات کلیدی: قضیه ایده ال اصلی تعمیم یافته، زیر مدول اول، زیر مدول کاملاً اول.