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$\mathsf{H}_v\mathsf{MV}$-ALGEBRAS II

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ABSTRACT. In this paper, we continue our study on HvMV-algebras. The quotient structure of an HvMV-algebra by a suitable type of congruence is studied, and some properties and related results are given. Some homomorphism theorems are given, as well. Also the fundamental HvMV-algebra, and the direct product of a family of HvMV-algebras are investigated, and some related results are obtained.

1. Introduction

In 1958, Chang [2] introduced the concept of an MV-algebra as an algebraic proof of completeness theorem for $\aleph_0$-valued Łukasiewicz propositional calculus; also see [3]. Many mathematicians have worked on MV-algebras, and obtained significant results. The hyperstructure theory (also called multialgebras) was introduced in 1934 by Marty [9]. Around the 40's, several authors worked on the hypergroups, especially in France and in the United States, but also in Italy, Russia, and Japan.

Recently, Ghorbani et al. [7] have applied the hyperstructures to MV-algebras, introduced the concept of hyper MV-algebra and investigated some related results; also see [8, 10]. Hyperstructures have many applications to several sectors of both the pure and applied sciences. A short review of the theory of hyperstructures has appeared in [4]. In [5], a wealth of applications can also be found. There are applications

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to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy set and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilities.

H_v-structures were introduced by Vougiouklis in the 4th AHA congress [11]; also see [12] and [13]. The concept of H_v-structure constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule, and so on). Actually, some axioms concerning the above hyperstructures such as the associative law, and distributive law have been replaced by their corresponding weak axioms. The reader finds in [12] some basic definitions and theorems about H_v-structures. Since then, the study of the H_v-structure theory has been pursued in many directions by Vougiouklis, Davvaz, Spartalis and others. A survey of the most results and applications of H_v-structure theory is based up on many papers, some of which contain more detailed presentations (see [6]).

In this paper, the quotient structure of $\mathsf{H}_v\mathsf{MV}$-algebras, direct product of $\mathsf{H}_v\mathsf{MV}$-algebras, and their direct product are introduced, and their properties are investigated, as mentioned in the abstract.

2. Preliminaries

This section is devoted to give some preliminaries from the literature For more details, we refer to the references.

Definition 2.1. An MV-*algebra* is an algebra $(M; +, ^*, 0)$ of type (2,1,0), satisfying the following properties:

(MV1) $+$ is associative,
(MV2) $+$ is commutative,
(MV3) $x + 0 = x$,
(MV4) $(x^*)^* = x$,
(MV5) $x + 0^* = 0^*$,
(MV6) $(x^* + y)^* + y = (y^* + x)^* + x$.

On any MV-algebra M, a binary relation '$\leq$' can be defined as $x \leq y$, if and only if $x^* + y = 0^*$. Then $\leq$ is a partial ordering in M.

In this section, the concept of an $\mathsf{H}_v\mathsf{MV}$-algebra is introduced, and some basic results are given.

Definition 2.2. An $\mathsf{H}_v\mathsf{MV}$-*algebra* is a non-empty set, H endowed with a binary hyperoperation '$\oplus$', a unary operation '*', and a constant, '0' satisfying the following conditions:

(H_vMV1) $x \oplus (y \oplus z) \cap (x \oplus y) \oplus z \neq \emptyset$, (weak associativity)
(H_vMV2) $x \oplus y \cap y \oplus x \neq \emptyset$, (weak commutativity)
(H_vMV3) $(x^*)^* = x$,

(H_vMV4) $(x^* \oplus y)^* \oplus y \cap (y^* \oplus x)^* \oplus x \neq \emptyset$,
(H_vMV5) $0^* \in x \oplus 0^* \cap 0^* \oplus x$,
(H_vMV6) $0^* \in x \oplus x^* \cap x^* \oplus x$,
(H_vMV7) $x \in x \oplus 0 \cap 0 \oplus x$,
(H_vMV8) $0^* \in x^* \oplus y \cap y \oplus x^*$ and $0^* \in y^* \oplus x \cap x \oplus y^*$ imply $x = y$.

Remark 2.3. On any H_vMV-algebra H, we can define a binary relation '$\preceq$' by

$$x \preceq y \ \Leftrightarrow \ 0^* \in x^* \oplus y \cap y \oplus x^*.$$

Hence, the condition (H_vMV8) can be redefined as follows:

$$x \preceq y \text{ and } y \preceq x \text{ imply } x = y.$$

Let A and B be non-empty subsets of H. By $A \preceq B$, we mean that there exist $a \in A$, and $b \in B$, such that $a \preceq b$. For $A \subseteq H$, denote the set $\{a^* : a \in A\}$ by A^* and 0^* by 1.

On H, we define a hyperoperation '$\odot$' as $x \odot y = (x^* \oplus y^*)^*$. The next theorem gives some properties.

Proposition 2.4. *In any H_vMV-algebra H, the following hold: $\forall x, y \in H$ and $\forall A, B \subseteq H$,*

(1) $x \odot (y \odot z) \cap (x \odot y) \odot z \neq \emptyset$,
(2) $x \odot y \cap y \odot x \neq \emptyset$,
(3) $0 \in x \odot 0 \cap 0 \odot x$,
(4) $0 \in x \odot x^* \cap x^* \odot x$,
(5) $x \in x \odot 1 \cap 1 \odot x$,

Definition 2.5. Let $(H; \oplus, ^*, 0_H)$, and $(K; \otimes, ^\star, 0_K)$ be H_vMV-algebras, and let $f : H \longrightarrow K$ be a function satisfying the following conditions:

(1) $f(0_H) = 0_K$,
(2) $f(x^*) = f(x)^\star$,
(3) $f(x^*) \preceq f(x)^\star$,
(4) $f(x \oplus y) = f(x) \otimes f(y)$,
(5) $f(x \oplus y) \subseteq f(x) \otimes f(y)$.

f is called a *homomorphism,* if it satisfies the conditions (1), (2), and (4); it is called a *weak homomorphism* if it satisfies the conditions (1), (3), and (5), Clearly, if f is a homomorphism, $f(1) = 1$. For convenience, we use the same operations for H and K.

By ker$f$, we mean the set $\{x \in H : f(x) = 0\}$. As usual, a homomorphism that is one-to-one (resp. onto) is called a *monomorphism* (resp. *epimorphism*). A homomorphism which is both an epimorphism and a monomorphism is called an *isomorphism.* If $f : H \longrightarrow K$ is an isomorphism, we say that H and K are isomorphic, and we write $H \simeq K$.

Theorem 2.6. *Let $f : H \longrightarrow K$ be a homomorphism.*

1. *f is one-to-one, if and only if $ker f = \{0\}$.*
2. *f is an isomorphism, if and only if there exists a homomorphism $f^{-1} : K \longrightarrow H$, such that $ff^{-1} = 1_K$ and $f^{-1}f = 1_H$.*

Definition 2.7. A non-empty subset S of H is called an $\mathsf{H}_v\mathsf{MV}$-*subalgebra* of H, if $(S; \oplus, ^*, 0)$ is itself an $\mathsf{H}_v\mathsf{MV}$-algebra.

The next proposition gives an equivalent condition for an $\mathsf{H}_v\mathsf{MV}$-subalgebra.

Proposition 2.8. *A nonempty subset S of H is an $\mathsf{H}_v\mathsf{MV}$-subalgebra of H if and only if*

1. *$x \oplus y \subseteq S$, for all $x, y \in S$,*
2. *$x^* \in S$, for all $x \in S$.*

Corollary 2.9. *A non-empty subset S of H is an $\mathsf{H}_v\mathsf{MV}$-subalgebra, if and only if*

1. *$0 \in S$,*
2. *$x^* \oplus y \subseteq S$, for all $x, y \in S$.*

Definition 2.10. Let I be a non-empty subset of H, satisfying
(I_0). $x \preceq y$, and $y \in I$ imply $x \in I$.
Then I is called

1. an $\mathsf{H}_v\mathsf{MV}$-*ideal*, if $x \oplus y \subseteq I$, for all $x, y \in I$,
2. a *weak* $\mathsf{H}_v\mathsf{MV}$-*ideal*, if $x \oplus y \preceq I$, for all $x, y \in I$.

It is easy to see that, in any $\mathsf{H}_v\mathsf{MV}$-algebra H, $\{0\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, and obviously, H is an $\mathsf{H}_v\mathsf{MV}$-ideal of H. Also, every $\mathsf{H}_v\mathsf{MV}$-ideal is a weak $\mathsf{H}_v\mathsf{MV}$-ideal.

Theorem 2.11. *Let $f : H \longrightarrow K$ be a homomorphism.*

1. *$ker f$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.*
2. *If I is an $\mathsf{H}_v\mathsf{MV}$-ideal of K, $f^{-1}(I)$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

From now on, in this paper, H is denoted by an $\mathsf{H}_v\mathsf{MV}$-algebra, unless otherwise stated.

3. Quotient structures

In this section, it is shown that how we can construct the quotient $\mathsf{H}_v\mathsf{MV}$-algebra from the old one, and some homomorphism theorems are stated and proved. We start with a definition.

Definition 3.1. Let θ be a binary relation in H, and $A, B \subseteq H$. We say that

(1) $A\theta_s B$, if for all $a \in A$, and for all $b \in B$, $a\theta b$,
(2) $A\theta B$, if for all $a \in A$, there exists $b \in B$, and for all $b \in B$, there exists $a \in A$, such that $a\theta b$,
(3) $A\theta_{sw} B$, if for all $a \in A$ there exists $b \in B$, such that $a\theta b$,
(4) $A\theta_w B$, if there exist $a \in A$, and $b \in B$, such that $a\theta b$.

Obviously, $\theta_s \subseteq \theta \subseteq \theta_{sw} \subseteq \theta_w$.

It must be noticed that when A and B are singleton, $\theta = \theta_s = \theta_{sw} = \theta_w$.

Proposition 3.2. *Let θ be a transitive relation in H, and $A, B, C \subseteq H$.*

(1) *If $A\theta_w B$ and $B\theta_{sw} C$, then $A\theta_w C$.*
(2) *If $A\theta_{sw} B$ and $B\theta_{sw} C$, then $A\theta_{sw} C$.*
(3) *If $A\theta_s B$ and $B\theta_s C$, then $A\theta_s C$.*
(4) *If $A\theta_s B$ and $B\theta_{sw} C$, then $A\theta_{sw} C$.*
(5) *If $A\theta_s B$ and $B\theta_w C$, then $A\theta_{sw} C$.*
(6) *If $A\theta_s B$ and $B\theta C$, then $A\theta C$.*
(7) *If $A\theta B$ and $B\theta C$, then $A\theta C$.*

Proof. Routine. □

Definition 3.3. Let θ be a binary relation in H with the property

$$x\theta y \text{ implies that } x^*\theta y^*. \tag{3.1}$$

θ is said to be

- *strongly compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \ \theta_s y \oplus v$.
- *compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \ \theta y \oplus v$.
- *s-weak compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \ \theta_{sw} y \oplus v$.
- *weakly compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \ \theta_w y \oplus v$.

It is clear that every strongly compatible relation is compatible, every compatible relation is s-weak compatible, and every s-weak compatible relation is weakly compatible.

Theorem 3.4. *Let θ be a reflexive and transitive binary relation in H. Then θ is compatible, if and only if (3.1) holds, and,*

$$x\theta y \text{ implies that } x \oplus a \ \theta \ y \oplus a \text{ and } a \oplus x \ \theta \ a \oplus y, \tag{3.2}$$

for all $x, y, a \in H$.

Proof. Assume that θ is compatible, $x\theta y$, and $a \in H$. Since θ is reflexive, so $a\theta a$, whence $x \oplus a \ \theta \ y \oplus a$ and $a \oplus x \ \theta \ a \oplus y$.

Conversely, assume that θ satisfies (3.2), $x\theta y$, and $u\theta v$. Hence, $x \oplus u \ \theta \ y \oplus u$ and $y \oplus u \ \theta \ y \oplus v$, whence, by Proposition 3.2(7), $x \oplus u \ \theta \ y \oplus v$. □

Remark 3.5. In virtue of Proposition 3.2, it is easy to see that an analogous result holds for strongly compatible relations, and s-weak compatible relations.

Proposition 3.6. *Every reflexive weakly compatible relation θ in H satisfies:*

$$x\theta y \text{ implies that } x \oplus a\ \theta_w\ y \oplus a \text{ and } a \oplus x\ \theta_w\ a \oplus x. \tag{3.3}$$

Proof. The proof is similar to the proof of Theorem 3.4. □

Proposition 3.7. *Let θ be a symmetric binary relation in H. Then θ is compatible, if and only if it is s-weak compatible.*

Proof. In virtue of the observation just after Definition 3.3, it is enough to prove that every symmetric s-weak compatible relation is compatible. Assume that θ is a symmetric s-weak compatible, and $x\theta y$ and $u\theta v$, for $x, y, u, v \in H$. Then $x \oplus u\ \theta_{sw}\ y \oplus v$, which means that for all $a \in x \oplus u$, there exists $b \in y \oplus v$, such that $a\theta b$. Since θ is symmetric, so $y\theta x$ and $v\theta u$, whence $y \oplus v\ \theta_{sw} x \oplus u$, i.e., for all $b \in y \oplus v$, there exists $a \in x \oplus u$, such that $a\theta b$. This implies that $x \oplus u\ \theta\ y \oplus v$, i.e., θ is compatible. □

In virtue of Definition 3.3 and Proposition 3.7, we define three types of congruences in H.

Definition 3.8. Let θ be an equivalence relation in H that satisfies (3.1). θ is called a:

- *strong congruence*, if it is strongly compatible.
- *congruence*, if it is compatible.
- *weak congruence*, if it is weakly compatible.

Corollary 3.9. *Let θ be an equivalence relation in H. θ is a congruence, if and only if it satisfies (3.1) and (3.2).*

Example 3.10. (i) Obviously, in any $\mathsf{H}_v\mathsf{MV}$-algebra H, ∇_H is a strong congruence in H.
(ii) Let $H = \{0, a, b, 1\}$, and let the operations $\oplus$ and * be defined as shown in Table 1. Then $(H; \oplus, ^*, 0)$ is a proper $\mathsf{H}_v\mathsf{MV}$-algebra (see [1]).

Let $\theta = \{(0, 0), (a, a), (b, b), (1, 1), (a, b), (b, a)\}$. Obviously, θ is an equivalence relation in H, which satisfies (3.1). Also it is easily verified that θ is weakly compatible. Hence, θ is a weak congruence in H.
(iii) Let $H = \{0, a, b, c, 1\}$, and consider Table 2. Then $(H; \oplus, ^*, 0)$ is an $\mathsf{H}_v\mathsf{MV}$-algebra (see [1]). Let

$$\theta = \{(0,0), (a,a), (b,b), (c,c), (1,1), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (0,1), (1,0)\}.$$

$\oplus$	0	a	b	1
0	{0,a}	{0,a,b}	{0,a,b}	{0,a,b,1}
a	{0,a,b,1}	{0,b}	{0,1}	{a,b,1}
b	{a,b}	{0,a,b,1}	{0}	{0,a,b,1}
1	{0,a,1}	{0,a,b,1}	{1}	{0,a,b,1}
$*$	1	b	a	0

TABLE 1. The Cayley table of $\oplus$ and *

$\oplus$	0	a	b	c	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,a,b,c,1}
a	{0,a}	{0,a}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}
b	{0,b}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c}	{0,a,b,c,1}
c	{0,c}	{0,a,b,c,1}	{0,a,b,c}	{0,a,b,c,1}	{0,a,b,c,1}
1	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}
$*$	1	b	a	c	0

TABLE 2. The Cayley table of '$\oplus$' and '*'

It is not difficult to verify that θ is a congruence in H.

In virtue of Remark 3.5, we can check that an analogous result holds for strong congruences and weak congruences.

Definition 3.11. A binary relation θ in H is called *regular*, if $x^* \oplus y\theta_w\{0^*\}$ and $y^* \oplus x\theta_w\{0^*\}$ imply $x\theta y$.

For a congruence θ in H, let x/θ be the congruence class of x, and $H/\theta = \{x/\theta : x \in H\}$. We define the operations '$\oplus$' and '*' on H/θ by

$$x/\theta \oplus y/\theta = \{a/\theta : a \in x \oplus y\} \text{ and } (x/\theta)^* = x^*/\theta.$$

Then we have the following theorem:

Theorem 3.12. *Let H be an H_vMV-algebra, and θ be a regular congruence in H. Then $(H/\theta, \oplus, ^*, 0/\theta)$ forms an H_vMV-algebra.*

Proof. We first prove that '$\oplus$' and '$*$' are well-defined. Let $x, y \in H$ be such that $x/\theta = y/\theta$. This implies that $x\theta y$, and so $x^*\theta y^*$, whence $x^*/\theta = y^*/\theta$. This means that $(x/\theta)^* = (y/\theta)^*$. Let $x_1, x_2, y_1, y_2 \in H$ be such that $x_1/\theta = y_1/\theta$, and $x_2/\theta = y_2/\theta$. Then $x_1 \oplus x_2 \ \theta \ y_1 \oplus x_2$, and $y_1 \oplus x_2 \ \theta \ y_1 \oplus y_2$ whence $x_1 \oplus x_2 \ \theta \ y_1 \oplus y_2$, by Proposition 3.2(7). If $a/\theta \in x_1/\theta \oplus x_2/\theta$, then $a/\theta = b/\theta$, for some $b \in x_1 \oplus x_2$ and so $a\theta b$ and $b\theta c$, where $c \in y_1 \oplus y_2$. Thus $a/\theta = c/\theta \in y_1/\theta \oplus y_2/\theta$, proving

$x_1/\theta \oplus x_2/\theta \subseteq y_1/\theta \oplus y_2/\theta$. In a similar way, we can prove that the converse inclusion holds. Thus $\oplus$ is well-defined.

The proof of the properties ($\mathsf{H}_v\mathsf{MV}1$)-($\mathsf{H}_v\mathsf{MV}7$) follows directly. The proof of ($\mathsf{H}_v\mathsf{MV}8$) follows from the regularity. □

Theorem 3.13. *If θ is a regular congruence in H, $0/\theta$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

Proof. Let $x, y \in H$ be such that $x \preceq y$, and $y \in 0/\theta$. Then $0^* \in x^* \oplus y \cap y \oplus x^*$ and $y/\theta = 0/\theta$, whence:

$$0^*/\theta \in x^*/\theta \oplus y/\theta \cap y/\theta \oplus x^*/\theta = x^*/\theta \oplus 0/\theta \cap 0/\theta \oplus x^*/\theta.$$

Hence, $x/\theta \preceq 0/\theta$, and so $x/\theta = 0/\theta$ means that $x \in 0/\theta$.

Now, let $x, y \in 0/\theta$. Then $x\theta 0$ and $0\theta y$, and so $x \oplus y \ \theta \ 0 \oplus y$, and $0 \oplus y \ \theta \ 0 \oplus 0$, whence $x \oplus y\theta 0 \oplus 0$. Since $0 \in 0 \oplus 0$, so there exists $a \in x \oplus y$, such that $a\theta 0$, i.e., $a \in 0/\theta$, and so $x \oplus y \cap 0/\theta \neq \emptyset$, whence $x \oplus y \preceq 0/\theta$, proving that $0/\theta$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

Open Problem 3.14. Let I be a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H. Is there a congruence θ in H, such that $0/\theta = I$?

The next theorem is easily proved, and so the proof is omitted.

Theorem 3.15. *If θ is a regular congruence in H, the mapping $\natural : H \longrightarrow H/\theta$ with $\natural(x) = x/\theta$ is an epimorphism with $ker\natural = 0/\theta$.*

The mapping $\natural$ is called the *canonical epimorphism.*

Theorem 3.16. *If θ is a regular congruence in H, and $f : H \longrightarrow K$ is a homomorphism of $\mathsf{H}_v\mathsf{MV}$-algebras, such that $0/\theta \subseteq kerf$, there exists a unique homomorphism $\bar{f} : H/\theta \longrightarrow K$, such that $\bar{f}(a/\theta) = f(a)$, for all $a \in H$, $Im\bar{f} = Imf$, and $ker\bar{f} = kerf/\theta$. $\bar{f}$ is an isomorphism, if and only if f is onto and $kerf = 0/\theta$.*

Proof. We first prove that $\bar{f}$ is well-defined. Let $a, b \in H$ be such that $a/\theta = b/\theta$. Then $0^*/\theta \in a^*/\theta \oplus b/\theta \cap b/\theta \oplus a^*/\theta$. This implies that $x/\theta = 0^*/\theta = y/\theta$, for some $x \in a^* \oplus b$ and $y \in b \oplus a^*$ whence $x^*, y^* \in 0/\theta \subseteq kerf$, i.e., $f(x^*) = f(0) = f(y^*)$. Hence,

$$0^* = f(0^*) = f(x) \in f(a)^* \oplus f(b)$$

and similarly, $0^* \in f(b) \oplus f(a)^*$, and hence, $f(a) \preceq f(b)$. In a similar way, we can show that $f(b) \preceq f(a)$. Thus $\bar{f}(a/\theta) = f(a) = f(b) = \bar{f}(b/\theta)$, i.e., $\bar{f}$ is well-defined. Obviously, $\bar{f}$ is a homomorphism, and $Im\bar{f} = Imf$. Now,

$$a/\theta \in ker\bar{f} \Rightarrow f(a) = \bar{f}(a/\theta) = 0 \Rightarrow a \in kerf \Rightarrow a/\theta \in kerf/\theta,$$

whence $ker\bar{f} \subseteq kerf/\theta$. Conversely, if $a/\theta \in kerf/\theta$, so $a/\theta = b/\theta$, for some $b \in kerf$, and hence, $f(a) = \bar{f}(a/\theta) = \bar{f}(b/\theta) = f(b) = 0$, i.e., $a \in kerf$. This implies that $kerf/\theta \subseteq ker\bar{f}$, proving $ker\bar{f} = kerf/\theta$. $\bar{f}$ is unique, because it is determined completely by f.

Finally, $\bar{f}$ is an isomorphism, if and only if it is an epimorphism and a monomorphism. Obviously, $\bar{f}$ is an epimorphism, if and only if f is an epimorphism, and by Theorem 2.6, $\bar{f}$ is a monomorphism, if and only if $kerf/\theta = ker\bar{f} = \{0/\theta\}$, i.e., if and only if $kerf = 0/\theta$. □

Corollary 3.17. *(Fundamental Homomorphism Theorem) Let θ be a regular congruence in H. Then every homomorphism $f : H \longrightarrow K$ of H_vMV-algebras induces an isomorphism $H/\theta \simeq Imf$, where $0/\theta = kerf$.*

Proof. Since $f : H \longrightarrow Imf$ is an epimorphism, and $kerf = 0/\theta$, so, by Theorem 3.16, the mapping $\bar{f} : H/\theta \longrightarrow Imf$ with $a/\theta \mapsto f(a)$ is an isomorphism. □

Corollary 3.18. *Let θ and ϑ be the regular congruences in H_vMV-algebras H and K, respectively, and let $f : H \longrightarrow K$ be a homomorphism with $f(0/\theta) \subseteq 0/\vartheta$. Then f induces a homomorphism $\bar{f} : H/\theta \longrightarrow K/\vartheta$ with $\bar{f}(a/\theta) = f(a)/\vartheta$. $\bar{f}$ is an isomorphism, if and only if $Imf/\vartheta = K$, and $f^{-1}(0/\vartheta) \subseteq 0/\theta$.*

Proof. Obviously, the composition $H \xrightarrow{f} K \xrightarrow{\natural} K/\vartheta$ is a homomorphism, and $0/\theta \subseteq f^{-1}(0/\vartheta) = ker\natural f$ because:

$$\begin{aligned} x \in ker\natural f &\Leftrightarrow \natural f(x) = 0/\vartheta \Leftrightarrow f(x)/\vartheta = 0/\vartheta \Leftrightarrow f(x) \in 0/\vartheta \\ &\Leftrightarrow x \in f^{-1}(0/\vartheta). \end{aligned}$$

Now, by Theorem 3.16, for $\natural f$ instead of f, and K/ϑ instead of K, the mapping $H/\theta \longrightarrow K/\vartheta$ with $a/\theta \mapsto (\natural f)(a) = f(a)/\vartheta$ is a homomorphism, which is an isomorphism, if and only if $\natural f$ is an epimorphism and $ker\natural f = 0/\theta$. But $ker\natural f = 0/\theta$, if and only if $f^{-1}(0/\vartheta) \subseteq 0/\theta$. Now, assume that $\natural f$ is an epimorphism, and $x \in K$. Then $x/\vartheta \in K/\vartheta$, and so $x/\vartheta = \natural f(h) = f(h)/\vartheta$, where $h \in H$. This implies that $x \in f(h)/\vartheta$, whence $K \subseteq Imf/\vartheta$. Obviously, $Imf/\vartheta \subseteq K$. Hence, $K = Imf/\vartheta$. Conversely, assume that $K = Imf/\vartheta$, and $x/\vartheta \in K/\vartheta$. Then $x \in K$, and so $x/\vartheta = f(a)/\vartheta = \natural f(a)$, for some $a \in H$, proving $\natural f$ is onto. Particulary, if f is onto, $Imf = K$, and so $Imf/\vartheta = K$. □

Let θ and ϑ be regular congruences in H such that $\vartheta \subseteq \theta$. Define a binary relation θ/ϑ in H/ϑ by

$$a/\vartheta(\theta/\vartheta)b/\vartheta \Leftrightarrow a\theta b.$$

It is obvious that θ/ϑ is a regular congruence in H/ϑ.

Corollary 3.19. *Let θ and ϑ be regular congruences in H and $\vartheta \subseteq \theta$. Then $(H/\vartheta)/(\theta/\vartheta) \simeq H/\theta$.*

Proof. Obviously, the mapping $f : H/\vartheta \longrightarrow H/\theta$ with $f(a/\vartheta) = a/\theta$ is an epimorphism, and

$$\begin{aligned} ker f &= \{a/\vartheta : a/\theta = f(a/\vartheta) = 0/\theta\} = \{a/\vartheta : a\theta 0\} \\ &= \{a/\vartheta : a/\vartheta(\theta/\vartheta)0/\vartheta\} \\ &= 0/(\theta/\vartheta) \end{aligned}$$

whence, by Corollary 3.17, $(H/\vartheta)/(\theta/\vartheta) \simeq Imf = H/\theta$. □

4. Fundamental MV-algebras

In this section, we introduce the concept of fundamental relation on $\mathsf{H}_v\mathsf{MV}$-algebras. We first give an application of strong congruences.

Theorem 4.1. *If θ is a regular strong congruence in H, $(H/\theta, \oplus, ^*, 0/\theta)$ is an* MV*-algebra.*

Proof. In virtue of Theorem 3.12, it is enough to prove that, for all $x, y \in H$, the set $x/\theta \oplus y/\theta$ is singleton. Let $a/\theta, b/\theta \in x/\theta \oplus y/\theta$. Then there exist $c, d \in x \oplus y$, such that $a/\theta = c/\theta$ and $b/\theta = d/\theta$. Since θ is reflexive, so $x\theta x$ and $y\theta y$, whence $x \oplus y\theta_s x \oplus y$. This implies that $c\theta d$, i.e., $a/\theta = c/\theta = d/\theta = b/\theta$, proving $|x/\theta \oplus y/\theta| = 1$. □

Let $\mathcal{U}_{ps}$ and $\mathcal{U}_{sp}$ be the set of all finite sums of finite products and the set of all finite products of finite sums of the elements of H, respectively, and let $\mathcal{U} = \mathcal{U}_{ps} \cup \mathcal{U}_{sp}$. Define a binary relation γ in H by

$$a\gamma b, \quad \text{if and only if } \{a, b\} \subseteq u, \text{ for some } u \in \mathcal{U}.$$

It is obvious that γ is reflexive and symmetric.

Define a binary relation γ^* in H by $a\gamma^* b$, if and only if there exist $n \in \mathbb{N}$, and $z_1, z_2, \ldots, z_{n+1} \in H$, such that $a = z_1$, $b = z_{n+1}$, and for all $i \in \{1, 2, \ldots, n\}$, $\{z_i, z_{i+1}\} \subseteq u_i$, for some $u_i \in \mathcal{U}$.

Theorem 4.2. *The relation γ^* is an equivalence relation in H.*

Proof. Let $a \in H$. From $a \in a \odot 1$, for $n = 2$, $z_1 = z_2 = a$, and $u = (0 \odot 1) \oplus (a \odot 1)$, we get

$$\{a\} \subseteq a \odot 1 \subseteq 0 \oplus (a \odot 1) \subseteq (0 \odot 1) \oplus (a \odot 1) = u,$$

whence $a\gamma^* a$, i.e., γ^* is reflexive. Now, let $a, b \in H$ be such that $a\gamma^* b$. Then there exist $n \in \mathbb{N}$, $z_1, z_2, \ldots, z_{n+1} \in H$, such that $a = z_1$, $b = z_{n+1}$, and for all $i \in \{1, 2, \ldots, n\}$, $\{z_i, z_{i+1}\} \subseteq u_i$, where $u_i \in \mathcal{U}$.

Let $y_i = z_{n-i+2}$, for all $i \in \{1, 2, \ldots, n\}$. Then $y_1 = b$, $y_{n+1} = a$, and $\{y_i, y_{i+1}\} \subseteq v_i$, where $v_i = u_{n-i+1} \in \mathcal{U}$, proving $b\gamma^* a$, i.e. γ^* is symmetric. For transitivity, let $a\gamma^* b$ and $b\gamma^* c$, for $a, b, c \in H$. Then there exist $n, m \in \mathbb{N}$, $z_1, \ldots, z_{n+1}, w_1, \ldots, w_{m+1} \in H$, such that $a = z_1$, $z_{n+1} = b = w_1$, $c = w_{m+1}$, and $\{z_i, z_{i+1}\} \subseteq u_i$ and $\{w_j, w_{j+1}\} \subseteq v_j$, where $u_i, v_j \in \mathcal{U}$, for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2 \ldots, m\}$. Let $x_i = z_i$, for $i \in \{1, 2, \ldots, n\}$, and $x_i = w_j$, where $i = n + j$, for $j \in \{1, 2, \ldots, m\}$. Then $a = x_1$, $c = x_{m+1}$ and $\{x_i, x_{i+1}\} \subseteq r_i$, where $r_i \in \mathcal{U}$. Thus γ^* is an equivalence relation in H. □

Theorem 4.3. *γ^* is the smallest regular strong congruence in H with the property that H/γ^* is an* MV*-algebra.*

Proof. In virtue of Theorem 3.4 and Remark 3.5, to prove that γ^* is a strong congruence, it is enough to prove that (3.1) holds, and for all $x, y, a \in H$,

$$x\gamma^* y \text{ implies that } x \oplus a\gamma_s^* y \oplus a, \text{ and } a \oplus x\gamma_s^* a \oplus y.$$

Assume that $x\gamma^* y$, for $x, y \in H$. Then there exist $n \in \mathbb{N}$, $a_1, \ldots, a_{n+1} \in H$, such that $x = a_1$, $y = a_{n+1}$, and for all $i \in \{1, 2, \ldots, n\}$, $\{a_i, a_{i+1}\} \subseteq u_i$, for some $u_i \in \mathcal{U}$. This implies that $x^* = a_1^*$, $y^* = a_{n+1}^*$, and $\{a_i^*, a_{i+1}^*\} \subseteq u_i^* \in \mathcal{U}$, whence $x^*\gamma^* y^*$. Thus (3.1) holds. Also, if $a \in H$, for $s_i \in a_i \oplus a$, we have:

$$\{s_i, s_{i+1}\} \subseteq (a_i \oplus a) \cup (a_{i+1} \oplus a) \subseteq u_i \oplus a \subseteq u_i \oplus (a \odot 1) = v_i \in \mathcal{U}$$

or $\{s_i, s_{i+1}\} \subseteq u_i \oplus a \subseteq u_i \odot (a \oplus 0) \in \mathcal{U}$. Thus for $s_1 \in x \oplus a = a_1 \oplus a$ and $s_{n+1} \in y \oplus a = a_{n+1} \oplus a$, we have $s_1 \gamma^* s_{n+1}$. This implies that $x \oplus a\gamma_s^* y \oplus a$. Similarly, we can show that $x\gamma^* y$ implies that $a \oplus x\gamma_s^* a \oplus y$. Thus γ^* is a strong congruence in H.

For regularity, assume that $x^* \oplus y\gamma_w^*\{0^*\}$, and $y^* \oplus x\gamma_w^*\{0^*\}$, for $x, y \in H$. Then $(x^* \oplus y)^*\gamma_w^*\{0\}$, and $(y^* \oplus x)^*\gamma_w^*\{0\}$, and so $(x^* \oplus y)^* \oplus y\gamma_s^* 0 \oplus y$, and $(y^* \oplus x)^* \oplus x\gamma_s^* 0 \oplus x$, whence $0 \oplus x\gamma_s^* 0 \oplus y$. Since $y \in 0 \oplus y$ and $x \in 0 \oplus x$, so $x\gamma^* y$ proving γ^* is regular. Therefore, γ^* is a regular strong congruence in H, and H/γ^* is an MV-algebra.

Now, let δ be a regular strong congruence in H with the property that H/δ is an MV-algebra, and $x\gamma y$, for $x, y \in H$. Then $\{x, y\} \subseteq u \in \mathcal{U}$. Assume that $u = \oplus_{i=1}^n (\odot_{j=1}^m x_{ij})$, where $x_{ij} \in H$. Since δ is a strong congruence, so $u/\delta = \oplus_{i=1}^n (\odot_{j=1}^m x_{ij}/\delta)$ is singleton, and since $x/\delta, y/\delta \in u/\delta$ implies that $x/\delta = y/\delta$, i.e., $x\delta y$. Hence, $\gamma \subseteq \delta$. Now, if $x\gamma^* y$, there exist $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_{n+1} \in H$, such that $x = a_1$, $y = a_{n+1}$ and $a_i \gamma a_{i+1}$, whence $a_i \delta a_{i+1}$. Since δ is transitive, so $x\delta y$, proving $\gamma^* \subseteq \delta$. Therefore, γ^* is the smallest regular strong congruence in H, such that H/γ^* is an MV-algebra. □

Remark 4.4. The relation γ^* is called the *fundamental relation* in H, and H/γ^* is called the *fundamental* MV*-algebra.*

5. Direct products

In this section, we define the direct product of a family of $\mathsf{H}_v\mathsf{MV}$-algebras, characterize the $\mathsf{H}_v\mathsf{MV}$-subalgebras and (weak) $\mathsf{H}_v\mathsf{MV}$-ideals of it, and give some homomorphism theorems.

Let $\{(H_i;\oplus_i,^{*_i},0_i) : i \in I\}$ be a non-empty family of $\mathsf{H}_v\mathsf{MV}$-algebras. The cartesian product $\prod_{i\in I} H_i$ of H_i's is defined as the set of all functions $f : I \longrightarrow \cup H_i$ with $f(i) \in H_i$, for all $i \in I$. For $f, g \in \prod_{i\in I} H_i$, define $f = g$, if and only if $f(i) = g(i)$, for all $i \in I$, and

$$f^*(i) = f(i)^{*_i} \text{ and } (f \oplus g)(i) = f(i) \oplus_i g(i), \forall i \in I.$$

Also define $0(i) = 0_i$, for all $i \in I$. It is easy to check that $\prod_{i\in I} H_i$ together with '$\oplus$', '*' satisfies ($\mathsf{H}_v\mathsf{MV}$1)-($\mathsf{H}_v\mathsf{MV}$8). Thus we get

Theorem 5.1. *If $\{(H_i;\oplus_i,^{*_i},0_i) : i \in I\}$ is a non-empty family of* $\mathsf{H}_v\mathsf{MV}$*-algebras;*

(1) $(\prod_{i\in I} H_i;\oplus,^*,0)$ *is an* $\mathsf{H}_v\mathsf{MV}$*-algebra,*
(2) *for each $k \in I$, the mapping $\pi_k : \prod_{i\in I} H_i \longrightarrow H_k$ with $f \mapsto f(k)$ is an epimorphism.*

$\prod_{i\in I} H_i$ is called the *direct product* of H_i's. If H_i is an $\mathsf{H}_v\mathsf{MV}$-algebra with the order $\preceq_i$, the order on $\prod_{i\in I} H_i$ is given by $f \preceq g$, if and only if $f(i) \preceq_i g(i)$.

The image of f can be written as $\{a_i\}$, where $a_i \in H_i$. In this case, the hyperoperation '$\oplus$' is written as $\{a_i\} \oplus \{b_i\} = \{a_i \oplus_i b_i\}$. If $I = \{1, 2, \ldots, n\}$ is finite, $\prod_{i\in I} H_i$ is written as $H_1 \times H_2 \times \cdots \times H_n$.

In the sequel, in this section, $\{H_i : i \in I\}$ is a non-empty family of $\mathsf{H}_v\mathsf{MV}$-algebras, and $\prod_{i\in I} H_i$ is the direct product of H_i's.

Theorem 5.2. *Let H_i be an* $\mathsf{H}_v\mathsf{MV}$*-algebra, and S_i be a non-empty subset of H_i with $i \in I$.*

(1) *If S_i is an* $\mathsf{H}_v\mathsf{MV}$*-subalgebra, $\prod_{i\in I} S_i$ is an* $\mathsf{H}_v\mathsf{MV}$*-subalgebra of $\prod_{i\in I} H_i$.*
(2) *If S_i is an* $\mathsf{H}_v\mathsf{MV}$*-ideal, $\prod_{i\in I} S_i$ is an* $\mathsf{H}_v\mathsf{MV}$*-ideal of $\prod_{i\in I} H_i$.*
(3) *If S_i is a weak* $\mathsf{H}_v\mathsf{MV}$*-ideal, $\prod_{i\in I} S_i$ is a weak* $\mathsf{H}_v\mathsf{MV}$*-ideal of $\prod_{i\in I} H_i$.*

Proof. Routine. □

Theorem 5.3. *Let H_i $(i \in I)$ be an* $\mathsf{H}_v\mathsf{MV}$*-algebra and S be a non-empty subset of $\prod_{i\in I} H_i$.*

(1) *If S is an $\mathsf{H}_v\mathsf{MV}$-subalgebra, there exists unique $\mathsf{H}_v\mathsf{MV}$-subalgebra S_i of H_i, for all $i \in I$, such that $S = \prod_{i\in I} S_i$.*
(2) *If S is an $\mathsf{H}_v\mathsf{MV}$-ideal, there exists unique $\mathsf{H}_v\mathsf{MV}$-ideal S_i of H_i, for all $i \in I$, such that $S = \prod_{i\in I} S_i$.*
(3) *If S is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, there exists unique weak $\mathsf{H}_v\mathsf{MV}$-ideal I_i of H_i, for all $i \in I$, such that $S = \prod_{i\in I} S_i$.*

Proof. We first observe that if S is a non-empty subset of $\prod_{i\in I} H_i$, for

$$S_i = \{a_i \in H_i : \exists f \in S, \text{ such that } f(i) = a_i\},$$

we get $\prod_{i\in I} S_i = S$.

(1) Assume that S is an $\mathsf{H}_v\mathsf{MV}$-subalgebra of $\prod_{i\in I} H_i$. We show that S_i is an $\mathsf{H}_v\mathsf{MV}$-subalgebra of H_i. Obviously, $S_i \neq \emptyset$ because $0_i \in S_i$, for all $i \in I$. Let $a_i, b_i \in S_i$, for $i \in I$. Then there exist $f, g \in S$, such that $f(i) = a_i$ and $g(i) = b_i$, whence $a_i^{*_i} \oplus_i b_i = f^{*_i}(i) \oplus_i g(i) = (f^* \oplus g)(i) \subseteq S_i$, proving S_i is an $\mathsf{H}_v\mathsf{MV}$-subalgebra of H_i.

Now, let T_i be an $\mathsf{H}_v\mathsf{MV}$-subalgebra of H_i, for all $i \in I$, such that $S = \prod_{i\in I} T_i$. We show that $T_i = S_i$, for all $i \in I$. Let $a_i \in H_i$. Then $a_i \in T_i$, if and only if there exists $f \in \prod_{i\in I} T_i = S = \prod_{i\in I} S_i$ such that $f(i) = a_i$, if and only if $a_i \in S_i$ means that $T_i = S_i$.

(2) By (1), S_i is closed with respect to $\oplus_i$, for all $i \in I$. Now, let $a_i \preceq_i b_i$ and $b_i \in S_i$. Let $f, g \in \prod_{i\in I} H_i$ be such that $f(i) = a_i$, and $g(i) = b_i$. Then:

$$\begin{aligned} 0_i^{*_i} \in a_i^{*_i} \oplus_i b_i \cap b_i \oplus_i a_i^{*_i} &= f^{*_i}(i) \oplus_i g(i) \cap g(i) \oplus_i f^{*_i}(i) \\ &= (f^* \oplus g)(i) \cap (g \oplus f^*)(i) \end{aligned}$$

whence $0^* \in f^* \oplus g \cap g \oplus f^*$, i.e. $f \preceq g \in \prod_{i\in I} S_i = S$. Since S is an $\mathsf{H}_v\mathsf{MV}$-ideal, so $f \in S$, and hence, $a_i = f(i) \in S_i$. The uniqueness is proved similar to the proof of (1).

(3) The proof is similar to the proof of (2). □

Definition 5.4. Let $\{H_i : i \in I\}$ be a non-empty family of $\mathsf{H}_v\mathsf{MV}$-algebras. The *weak direct product* of H_i's, denoted by $\prod_{i\in I}^w H_i$, is defined as the set of all $f \in \prod_{i\in I} H_i$, such that for all but a finite number of $i \in I$, we have $f(i) = 0_i$.

Remark 5.5. Note that when I is finite, the weak direct product and direct product are equal.

Theorem 5.6. *Let $\{H_i : i \in I\}$ be a non-empty family of $\mathsf{H}_v\mathsf{MV}$-algebras.*

(1) *If $0_i \oplus 0_i = \{0_i\}$, for all $i \in I$, $\prod_{i\in I}^w H_i$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of $\prod_{i\in I} H_i$.*

(2) *the mapping* $\iota_k : H_k \longrightarrow \prod_{i\in I}^{w} H_i$ *given by* $\iota_k(a) = \{a_i\}_{i\in I}$, *in which* $a_k = a$ *and* $a_i = 0_i$, *for all* $i \neq k$, *is a weak monomorphism.*

(3) *If* $0_i \oplus 0_i = \{0_i\}$, *for all* $i \in I$, $\iota_i(H_i)$ *is an* $\mathsf{H}_v\mathsf{MV}$*-ideal of* $\prod_{i\in I}^{w} H_i$.

Proof. (1) Let $f, g \in \prod_{i\in I} H_i$ be such that $f \preceq g$, and $g \in \prod_{i\in I}^{w} H_i$. Then $f(i) \preceq_i g(i)$ and $g(i) = 0_i$, for all but a finite number of $i \in I$ whence $0_i^{*_i} \in f^{*_i}(i) \oplus_i g(i) \cap g(i) \oplus_i f^{*_i}(i)$. Now, for $i \in I$ with $g(i) = 0_i$, we have $0_i^{*_i} \in f^{*_i}(i) \oplus_i 0_i \cap 0_i \oplus_i f^{*_i}(i)$, which implies that $f(i) \preceq_i 0_i$, i.e., $f(i) = 0_i$, proving $f \in \prod_{i\in I}^{w} H_i$. Let $f, g \in \prod_{i\in I}^{w} H_i$. Then $f(i) = 0_i$ and $g(j) = 0_j$, for all but a finite number of $i, j \in I$. Let $k \in I$ be the smallest element, for which $f(k) = g(k) = 0_k$. Then $(f \oplus g)(k) = f(k) \oplus_k g(k) = 0_k \oplus 0_k = \{0_k\}$, for all but a finite number of $k \in I$. Thus $f \oplus g \subseteq \prod_{i\in I}^{w} H_i$. Therefore, $\prod_{i\in I}^{w} H_i$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of $\prod_{i\in I} H_i$.

(2) Let $a \in H_k$. Then $\iota_k(a^{*_k}) = \{a_i\}_{i\in I}$ in which $a_k = a^{*_k}$ and $a_i = 0_i$, for all $i \neq k$. On the other hand, $\iota_k(a)^* = \{a_i\}_{i\in I}^* = \{a_i^{*_i}\}_{i\in I}$, in which $a_k^{*_k} = a^{*_k}$ and $a_i^{*_i} = 1$, for all $i \neq k$. This implies that $\iota_k(a^{*_k}) \preceq \iota_k(a)^{*_i}$. Now, let $a, b \in H_i$. Then:

$$\begin{aligned}
\iota_k(a \oplus_k b) &= \{\iota_k(c) : c \in a \oplus_k b\} \\
&= \{\{a_i\}_{i\in I} : a_k = c \in a \oplus_k b, a_i = 0_i, \forall i \neq k\} \\
&\subseteq \{a_i \oplus b_i\} \text{ with } a_k = a, b_k = b, a_i = b_i = 0_i, \forall i \neq k \\
&= \{a_i\}_{i\in I} \oplus \{b_i\}_{i\in I} \text{ with } a_k = a, b_k = b, a_i = b_i = 0_i, \forall i \neq k \\
&= \iota_k(a) \oplus \iota_k(b)
\end{aligned}$$

proving that ι_k is a weak homomorphism. Obviously, ι_k is one-to-one.

(3) Let $\{a_i\} \preceq \{b_i\}$, and $\{b_i\} \in \iota_k(H_k)$. Then $0^* \in \{a_i\}^* \oplus \{b_i\} \cap \{b_i\} \oplus \{a_i\}^*$, and $b_k = a \in H_k$ and $b_i = 0_i$, for all $i \neq k$, whence $0_i^{*_i} \in a_i^{*_i} \oplus_i b_i$, $b_k = a$, and $b_i = 0_i$, for all $i \neq k$. This implies that $0^{*_i} \in a_i^{*_i} \oplus_i 0_i$, for all $i \neq k$, and hence, $a_i \preceq 0_i$, i.e., $a_i = 0_i$, for all $i \neq k$. Thus $\{a_i\} \in \iota_k(H_k)$. Now, let $\{a_j\}, \{b_j\} \in \iota_i(H_i)$. Then $a_j = b_j = 0_j$, for all $j \neq i$, and so, $\{a_j\} \oplus \{b_j\} = \{a_j \oplus_j b_j\}$ in which $a_j \oplus_j b_j = 0_j \oplus_j 0_j = \{0_j\}$, proving $\{a_j\} \oplus \{b_j\} \subseteq \iota_i(H_i)$. □

Theorem 5.7. *Let* $\{f_i : H_i \longrightarrow K_i : i \in I\}$ *be a non-empty family of homomorphisms of* $\mathsf{H}_v\mathsf{MV}$*-algebras, and the mapping* $f = \prod f_i : \prod_{i\in I} H_i \longrightarrow \prod_{i\in I} K_i$ *be given by* $\{a_i\} \mapsto \{f_i(a_i)\}$. *Then* f *is a homomorphism such that* $f(\prod_{i\in I}^{w} H_i) \subseteq \prod_{i\in I}^{w} K_i$, $ker f = \prod_{i\in I} ker f_i$ *and* $Im f = \prod_{i\in I} Im f_i$. *Thus* f *is a monomorphism (resp. an epimorphism) if and only if so is each* f_i.

Proof. Routine. □

Theorem 5.8. *Let $\{H_i : i \in I\}$ be a non-empty family of $\mathsf{H}_v\mathsf{MV}$-algebras, and $\beta_i^{*_i}$ be the fundamental equivalence relation in H_i, for all $i \in I$, and β^* be the fundamental equivalence relation in $\prod_{i\in I} H_i$. Then $(\prod_{i\in I} H_i)/\beta^* \simeq \prod_{i\in I} H_i/\beta_i^{*_i}$.*

Proof. Consider the relation $\tilde{\beta}$ in $\prod_{i\in I} H_i$, as follows:

$$\{a_i\}\tilde{\beta}\{b_i\} \Leftrightarrow a_i\beta_i^{*_i}b_i, \forall i \in I.$$

Obviously, $\tilde{\beta}$ is a congruence relation in $\prod_{i\in I} H_i$. To prove regularity, let $\{a_i\}, \{b_i\} \in \prod_{i\in I} H_i$ be such that $\{a_i\}^* \oplus \{b_i\}\tilde{\beta}\{\{0_i\}\}$ and $\{b_i\}^* \oplus \{a_i\}\tilde{\beta}\{\{0_i\}\}$. Then $\{a_i^{*_i} \oplus_i b_i\}\tilde{\beta}\{\{0_i\}\}$ and $\{b_i^{*_i} \oplus_i a_i\}\tilde{\beta}\{\{0_i\}\}$, whence $\{c_i\}\tilde{\beta}\{0_i\}$ and $\{d_i\}\tilde{\beta}\{0_i\}$, for all $c_i \in a_i^{*_i} \oplus_i b_i$ and $d_i \in b_i^{*_i} \oplus_i a_i$. Hence, $c_i\beta_i^{*_i}0_i$ and $d_i\beta_i^{*_i}0_i$, i.e., $a_i^{*_i} \oplus_i b_i\beta_i^{*_i}\{0_i\}$ and $b_i^{*_i} \oplus_i a_i\beta_i^{*_i}\{0_i\}$, whence $a_i\beta_i^{*_i}b_i$, proving $\{a_i\}\tilde{\beta}\{b_i\}$.

Now, define '*' and '$\oplus$' on $(\prod_{i\in I} H_i)/\tilde{\beta}$ by

$$(\{a_i\}/\tilde{\beta})^* = \{a_i^{*_i}\}/\tilde{\beta},\ \{a_i\}/\tilde{\beta}\oplus\{b_i\}/\tilde{\beta} = \{\{c_i\}/\tilde{\beta} : c_i \in a_i/\beta^{*_i}\oplus_i b_i/\beta^{*_i}\}.$$

It is easy to check that $\tilde{\beta}$ is the smallest regular congruence relation in $\prod_{i\in I} H_i$, such that $(\prod_{i\in I} H_i)/\tilde{\beta}$ is an $\mathsf{H}_v\mathsf{MV}$-algebra, so $\tilde{\beta} = \beta^*$.

Now, by Theorem 3.15, the mapping $\natural_i : H_i \longrightarrow H_i/\beta_i^{*_i}$ is an epimorphism with $ker\natural_i = 0_i/\beta_i^{*_i}$ and so, by Theorem 5.7, $\prod \natural_i : \prod_{i\in I} H_i \longrightarrow \prod_{i\in I} H_i/\beta_i^{*_i}$ is an epimorphism with

$$ker(\prod \natural_i) = \prod_{i\in I} ker\natural_i = \prod_{i\in I} 0_i/\beta_i^{*_i} = 0/\beta^*.$$

Thus, by Corollary 3.17, $(\prod_{i\in I} H_i)/\beta^* \simeq \prod_{i\in I} H_i/\beta_i^{*_i}$. □

6. Conclusions and the future works

Based on the first work on $\mathsf{H}_v\mathsf{MV}$-algebras, in this paper, we introduced some types of congruences, studied the quotient $\mathsf{H}_v\mathsf{MV}$-algebra, and obtained some homomorphism theorems. Moreover, we obtained the fundamental equivalence relation on an $\mathsf{H}_v\mathsf{MV}$-algebra, the smallest equivalence relation on an $\mathsf{H}_v\mathsf{MV}$-algebra to make it to an MV-algebra. Finally, we introduced the direct product of a non-empty family of $\mathsf{H}_v\mathsf{MV}$-algebras. We proved that the direct product of a family of $\mathsf{H}_v\mathsf{MV}$-subalgebras ($\mathsf{H}_v\mathsf{MV}$-ideals, weak $\mathsf{H}_v\mathsf{MV}$-ideals) is again an $\mathsf{H}_v\mathsf{MV}$-subalgebra ($\mathsf{H}_v\mathsf{MV}$-ideal, weak $\mathsf{H}_v\mathsf{MV}$-ideal). Also we characterized the $\mathsf{H}_v\mathsf{MV}$-subalgebras ($\mathsf{H}_v\mathsf{MV}$-ideals, weak $\mathsf{H}_v\mathsf{MV}$-ideals) of the direct product of a family of $\mathsf{H}_v\mathsf{MV}$-algebras via the $\mathsf{H}_v\mathsf{MV}$-subalgebras ($\mathsf{H}_v\mathsf{MV}$-ideals, weak $\mathsf{H}_v\mathsf{MV}$-ideals) of any member of the family. Then using the fundamental equivalence relation, we obtained some homomorphism theorems.

The category of $\mathsf{H}_v\mathsf{MV}$-algebras and fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals could be the topics for further research works.

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H_vMV-ALGEBRAS II

M. BAKHSHI

H_vMV-جبرها ۲

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گروه ریاضی دانشگاه بجنورد

در این مقاله، تحقیق در مورد H_vMV-جبرها را ادامه می دهیم. به این ترتیب که ساختار خارج قسمتی آنها را مطالعه و برخی خواص و نتایج مرتبط را بدست می آوریم. همچنین چند قضیه همریختی را بیان و اثبات می کنیم. بعلاوه H_vMV-جبرهای اساسی را معرفی و نیز حاصلضرب مستقیم H_vMV-جبر ها را بررسی و نتایج مرتبط با آنها را ارائه می دهیم.

کلمات کلیدی: MV- جبر، H_vMV-جبر، MV-جبر اساسی.