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FUZZY NEXUS OVER AN ORDINAL

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ABSTRACT. In this paper, the fuzzy subnexuses over a nexus N are defined and the notions of prime fuzzy subnexuses and fractions induced by them are studied. Finally, it is shown that if S is a meet closed subset of the set $Fsub(N)$, of fuzzy subnexuses of a nexus N, and $h = \bigwedge S \in S$, then the fractions $S^{-1}N$ and $\{h\}^{-1}N$ are isomorphic as meet-semilattices.

1. Introduction

Fuzzy sets were introduced by Lotfi A. Zadeh [15] and Dieter Klaua [10] in 1965 as an extension of the classical notion of sets. At the same time, Salii [14] defined a more general kind of structures called L-relations, which were studied by him in an abstract algebraic context. Fuzzy relations, which are used now in different areas such as algebra [6, 12], rough set [4, 7], and clustering [3], are special cases of L-relations when L is the unit interval $[0, 1]$.

Section 2 of this paper is a prerequisite for the rest of the paper. The definitions and results of this section are taken from [2, 5, 8, 9, 11]. In Section 3, a fuzzy subnexus over an ordinal is defined, and also a prime fuzzy subnexus over an ordinal is defined. Particularly, we show that for every nexus N, and $f \in Fsub(N)$:

(1) If $|Imf| \leq 2$, and $\emptyset \neq f_* \in Psub(N)$, then $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$.
(2) If $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, for every $g, h \in Fsub(N)$, then $|Imf| \leq 2$.

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(3) If $|Imf| = 2$, and for every $g, h \in Fsub(N)$, $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, then $\emptyset \neq f_* \in Psub(N)$.

In Section 4, we introduce the notion fraction induced by fuzzy subnexuses, and give some characterizations for fraction of N in particular, we show that if S_1 and S_2 are meet closed subsets of $Fsub(N)$ and $h = \bigwedge S_1 = \bigwedge S_2 \in S_1 \cap S_2$, then $S_1^{-1}N \cong S_2^{-1}N \cong \{h\}^{-1}N$ as meet-semilattices.

2. Preliminaries

A partially ordered set A is a *meet-semilattice*, if the infimum for each pair of elements exists. A homomorphism is a function $f : N \to M$ between the meet-semilattices N and M, such that $f(x \wedge y) = f(x) \wedge f(y)$ for all x and y in N. Each *homomorphism* is order preserving, i.e. $x \leq y$ implies that $f(x) \leq f(y)$.

A subset D of poset A is *directed*, provided that it is non-empty, and every finite subset of D has an upper bound in D.

Let A be a poset. For $X \subseteq A$ and $x \in A$, we write:

(1) $\downarrow X = \{a \in A : a \leq x \text{ for some } x \in X\}$.
(2) $\uparrow X = \{a \in A : a \geq x \text{ for some } x \in X\}$.
(3) $\downarrow x = \downarrow \{x\}$.
(4) $\uparrow x = \uparrow \{x\}$.

We also say:

(5) X is a *lower set*, if and only if $X = \downarrow X$.
(6) X is an *upper set*, if and only if $X = \uparrow X$.
(7) X is an *ideal*, if and only if it is a directed lower set.
(8) An ideal is *principal*, if and only if it has a maximum element.

For undefined terms and notations, see [5, 11].

The collection of all ordinal numbers is a proper class, and we denote it as $\mathfrak{O}$. It is also customary to denote the order relation between ordinals by $\alpha < \beta$ instead of the two equivalent forms $\alpha \subset \beta$, $\alpha \in \beta$, though the latter is also quite common. If α is an ordinal, then, by definition, we have $\alpha = \{\beta \in \mathfrak{O} | \beta < \alpha\}$. If $\alpha, \beta \in \mathfrak{O}$, then either $\alpha < \beta$ or $\beta < \alpha$ or $\alpha = \beta$. If A is a set of ordinals, then $\bigcup A$ is an ordinal.

Let $\gamma, \delta \in \mathfrak{O}$, $\gamma \geq 1$, and $\delta \geq 1$. An *address* over γ is a function $a : \delta \to \gamma$ such that $a(i) = 0$ implies that $a(j) = 0$, for all $j \geq i$. We denote by $A(\gamma)$, the set of all addresses over γ.

Let $a : \delta \to \gamma$ be an address over γ. If, for every $i \in \delta$, $a(i) = 0$, then it is called the *empty address*, and denoted by $()$. If a is a non-empty address, then there exists a unique element $\beta \in \delta + 1$, such that, for every $i \in \beta$, $a(i) \neq 0$, and for every $\beta \leq i \in \delta$, $a(i) = 0$. We denote this address by $(a_i)_{i \in \beta}$, where $a_i = a(i)$ for every $i \in \beta$.

Let $a : \delta \to \gamma$, and $b : \beta \to \eta$ be addresses and $\delta \leq \beta$. We say $a = b$, if for every $i \in \delta$, $a_i = b_i$, and for every $i \in \beta \setminus \delta$, $b_i = 0$. In other words, there exists a unique element $\beta \in \mathfrak{O}$, such that $a = (a_i)_{i\in\beta} = b$.

The *level* of $a \in A(\gamma)$ is said to be:

(1) 0 , if $a = ()$.
(2) β, if $() \neq a = (a_i)_{i\in\beta}$.

The level of a is denoted by $l(a)$.

Let a and b be two elements of $A(\gamma)$. Then we say that $a \leq b$, if $l(a) = 0$ or one of the following cases satisfies for $a = (a_i)_{i\in\beta}$ and $b = (b_i)_{i\in\delta}$:

(1) If $\beta = 1$, then $a_0 \leq b_0$.
(2) If $\beta \geq 2$ is a non-limit ordinal, then $a|_{\beta-1} = b|_{\beta-1}$ and $a_{\beta-1} \leq b_{\beta-1}$.
(3) If β is a limit ordinal, then $a = b|_\beta$.

Proposition 2.1. [9] *$(A(\gamma), \leq)$ is a meet-semilattice.*

Let $() \neq a = (a_i)_{i\in\beta}$ be an element of $A(\gamma)$. For every $\delta \in \beta$ and $0 \leq j \leq a_\delta$, we put $a^{(\delta,j)} : \delta + 1 \to \gamma$, such that for every $i \in \delta + 1$,

$$a_i^{(\delta,j)} = \begin{cases} a_i & \text{if } i \in \delta; \\ j & \text{if } i = \delta. \end{cases}$$

Definition 2.2. [9] A *nexus* N over γ is a set of addresses with the following properties:

(1) $\emptyset \neq N \subseteq A(\gamma)$.
(2) If $() \neq a = (a_i)_{i\in\beta} \in N$, then for every $\delta \in \beta$ and $0 \leq j \leq a_\delta$, $a^{(\delta,j)} \in N$.

Proposition 2.3. [9] *Let N be the set of addresses over γ. Then, N is a nexus over γ, if and only if $\emptyset \neq N \subseteq A(\gamma)$, and for every $(a, b) \in N \times A(\gamma)$, $b \leq a$ implies that $b \in N$.*

Proposition 2.4. [9] *Let N be a nexus over γ. Then $(N, \leq)$ is a meet-semilattice.*

Let N be a nexus over γ, and $\emptyset \neq M \subseteq N$. Then M is called a *subnexus* of N, if M itself is a nexus over γ. The set of all subnexuses of N is denoted by $Sub(N)$. It is clear that $\{()\}$ and N are the trivial subnexuses of nexus N.

Proposition 2.5. [9] *If N is a nexus over γ, and $\{M_i\}_{i\in I} \subseteq Sub(N)$, then $\bigcup_{i\in I} M_i \in Sub(N)$ and $\bigcap_{i\in I} M_i \in Sub(N)$.*

Let N be a nexus over γ, and $X \subseteq N$. The smallest subnexus of N containing X is called the *subnexus of N generated* by X, and denoted by $< X >$. If $|X| = 1$, then $< X >$ is called a cyclic subnexus of N. It is clear that $< \emptyset >= \{()\}$, and $< N >= N$.

Remark 2.6. [9] Let $\emptyset \neq N \subseteq A(\gamma)$. Then, N is a nexus over γ, if and only if:

$$N =\downarrow N = \bigcup_{a\in N} \downarrow a.$$

A proper subnexus P of a nexus N over γ is said to be a *prime subnexus* of N if $a \wedge b \in P$ implies that $a \in P$ or $b \in P$, for every $a, b \in N$. The set of all prime subnexuses of N is denoted by $Psub(N)$.

Proposition 2.7. [9] *Let P be a proper subnexus of a nexus N over γ. Then, P is a prime subnexus of N, if and only if $N \setminus P$ is closed under finite meet.*

Corollary 2.8. [9] *Let N be a nexus over γ, and $\emptyset \neq X \subseteq N$. If X is closed under finite meet, then there exists $a \in X$, such that $\uparrow a =\uparrow X$, and $a = \bigwedge X$.*

A *fuzzy subset* f on set X is a function $f : X \to [0, 1]$. We denote by $F(X)$ the set of all fuzzy subsets of X. For $f, g \in F(X)$, we say $f \subseteq g$, if and only if $f(x) \leq g(x)$ for every $x \in X$. Let $f \in F(X)$, and $t \in [0, 1]$. Then the set $f_t = \{x \in X : f(x) \geq t\}$ is called the *level subset* of X with respect to f. Also we put $f_* = \{x \in X : f(x) = 1\}$. For $x \in X$ and $t \in (0, 1]$, $x^t \in F(X)$ is called a *fuzzy point*, if and only if $x^t(y) = 0$ for $y \neq x$ and $x^t(x) = t$. The fuzzy point x^t is said to belong to $f \in F(X)$, written $x^t \in f$, if and only if $f(x) \geq t$. If $f, g \in F(X)$, then $f \subseteq g$, if and only if $x^t \in f$ implies $x^t \in g$ for every fuzzy point $x^t \in F(X)$. For evrey $f, g \in F(X)$, and $r, s \in [0, 1]$, $(f \cap g)_r = f_r \cap g_r$, $(f \cup g)_r = f_r \cup g_r$, and if $r \leq s$, then $f_r \supseteq f_s$. For every $\{f_i\}_{i\in I} \subseteq F(X)$ and $r \in [0, 1]$, $\bigcup_{i\in I}(f_i)_r \subseteq (\bigcup_{i\in I} f_i)_r$ and $\bigcap_{i\in I}(f_i)_r = (\bigcap_{i\in I} f_i)_r$. For evrey $f, g \in F(X)$, $f \subseteq g \Leftrightarrow f_r \subseteq g_r$, for all $r \in [0, 1]$ (see [8]).

3. Prime fuzzy nexus

In this section, the notions of a fuzzy nexus and a prime fuzzy subnexus of a nexus are defined, and we discuss the relation subnexus and fuzzy subnexus, prime subnexus, and prime fuzzy subnexus.

Definition 3.1. Let f be a fuzzy subset on a nexus N. Then f is called a *fuzzy subnexus* of N, if $a \leq b$ implies that $f(b) \leq f(a)$ for all $a, b \in N$. The set of all fuzzy subnexuses of N is denoted by $Fsub(N)$.

Proposition 3.2. *Let A be a non-empty subset of a nexus N. Then, $A \in Sub(N)$, if and only if $\chi_A \in Fsub(N)$, where that χ_A is the characteristic function of A.*

Proof. Let $A \in Sub(N)$, and $a \leq b$, for some $a, b \in N$. If $b \in A$, by Proposition 2.3, $a \in A$, and so, $\chi_A(a) = \chi_A(b) = 1$. But if $b \notin A$, then $\chi_A(b) = 0$, and so, $\chi_A(b) \leq \chi_A(a)$, hence, $\chi_A \in Fsub(N)$.

Conversely, let $(a, b) \in A \times N$, and $b \leq a$. Then $1 = \chi_A(a) \leq \chi_A(b)$, which follows that $\chi_A(b) = 1$, i.e. $b \in A$. Hence, $A \in Sub(N)$. □

Proposition 3.3. *Let f be a fuzzy subset of N. Then $f \in Fsub(N)$, if and only if $f_r \in Sub(N)$, for every $r \in [0, 1]$, where $f_r \neq \emptyset$.*

Proof. Suppose $f \in Fsub(N)$ and $f_r \neq \emptyset$, for $r \in [0, 1]$, and let $b \in N, a \in f_r$, such that $b \leq a$. Then $f(b) \geq f(a) \geq t$, and hence, $b \in f_r$.

Conversly, suppose that f is a fuzzy subset of N, such that $f_r \in sub(N)$ for every $r \in [0, 1]$. Now let $a, b \in N$, $a \leq b$. We show that $f(b) \leq f(a)$. Let $f(b) = r$, for $r \in [0, 1]$. Thus $b \in f_r \neq \emptyset$, and since $f_r \in Sub(N)$, we can conclude from Proposition 2.3 that $a \in f_r$. Hence, $f(a) \geq r = f(b)$. □

Proposition 3.4. *Let N be a nexus over γ, and $\{f_i\}_{i \in I} \subseteq Fsub(N)$. Then:*

(1) $\bigcup_{i \in I} f_i \in Fsub(N)$.
(2) $\bigcap_{i \in I} f_i \in Fsub(N)$.

Proof. Let $a, b \in N$, and $a \leq b$ Then

$$(\bigcup_{i \in I} f_i)(b) = \bigvee_{i \in I} f_i(b) \leq \bigvee_{i \in I} f_i(a) = (\bigcup_{i \in I} f_i)(a)$$

and

$$(\bigcap_{i \in I} f_i)(b) = \bigwedge_{i \in I} f_i(b) \leq \bigwedge_{i \in I} f_i(a) = (\bigcap_{i \in I} f_i)(a).$$

□

Let N be a nexus over γ. For $f \in F(N)$, we put

$$< f >= \bigcap_{f \subseteq g \in Fsub(N)} g.$$

It is clear that $< f >$ is a fuzzy subnexus of N.

Proposition 3.5. *Let N be a nexus over γ, and f be a fuzzy subset of N Then:*

$$< f > (a) = \bigvee_{b \in \uparrow a} f(b).$$

Proof. Let f be a fuzzy subset of N. Define $h : N \longrightarrow [0, 1]$, with $h(a) = \bigvee_{b \in \uparrow a} f(b)$. We are going to show that h is the smallest fuzzy subnexus of N, which $f \subseteq h$. Let $a, b \in N$, and $a \leq b$. Since $\uparrow b \subseteq \uparrow a$, we can conclude that

$$h(a) = \bigvee_{z \in \uparrow a} f(z) \geq \bigvee_{z \in \uparrow b} f(z) = h(b).$$

Hence, $h \in Fsub(N)$. Now, let $g \in Fsub(N)$, $f \subseteq g$. Then for every $b \in \uparrow a$, we have $g(a) \geq f(b)$, which follows that $g(a) \geq \bigvee_{b \in \uparrow a} f(b)$. Hence, $g(a) \geq h(a)$, i.e. $h \subseteq g$. □

Proposition 3.6. *If N is a nexus over γ, and $f, g \in F(N)$, then*

$$< f > \cap < g > \geq < f \cap g > .$$

Proof. For every $a \in N$,

$$\begin{aligned} (< f > \cap < g >)(a) &= min\{< f > (a), < g > (a)\} \\ &= min\{\bigvee_{b \in \uparrow a} f(b), \bigvee_{b \in \uparrow a} g(b)\} \\ &\geq \bigvee_{b \in \uparrow a} min\{f(b), g(b)\} \\ &= \bigvee_{b \in \uparrow a} (f \cap g)(b) \\ &= < f \cap g > (a). \end{aligned}$$

Hence, $< f > \cap < g > \geq < f \cap g >$. □

Example 3.7. Let $\gamma = 3$, $N = \{(), (1), (2)\}$, and $f, g : N \to [0, 1]$ be functions such that

$$f = \begin{pmatrix} () & (1) & (2) \\ 0.1 & 0.2 & 0.3 \end{pmatrix}$$

and

$$g = \begin{pmatrix} () & (1) & (2) \\ 0.3 & 0.2 & 0.1 \end{pmatrix}.$$

It is clear that $< f > \cap < g > \neq < f \cap g >$.

Definition 3.8. Let N be a non-trivial nexus over γ, i.e. $N \neq \{()\}$. A fuzzy subnexus f of N is called a *prime fuzzy subnexus*, if

$$f(a \wedge b) \leq max\{f(a), f(b)\},$$

for all $a, b \in N$. The set of all prime fuzzy subnexuses of N is denoted by $PFsub(N)$.

It is clear that if $f \in PFsub(N)$, then $f(a \wedge b) = f(a)$ or $f(b)$, for all $a, b \in N$.

Proposition 3.9. *Let N be a non-trivial nexsus over γ, and f be a fuzzy subnexus of N. The following assertions are equivalent:*

(1) *f is a prime fuzzy subnexus.*
(2) *For every $r \in [0,1]$, if f_r is a non-empty subset N, then f_r is a prime subnexus of N.*
(3) *For every $r \in [0,1]$, $N \setminus f_r$ is closed under finite meet.*

Proof. (1) $\Rightarrow$ (2). Let $r \in [0,1]$, and f_r be a non-empty subset of N. If $a, b \in N$ and $a \wedge b \in f_r$, then $r \leq f(a \wedge b) \leq max\{f(a), f(b)\}$, and which follows that $a \in f_r$ or $b \in f_r$. By Proposition 3.3, f_r is a prime subnexus of N.

(2) $\Rightarrow$ (3). Suppose that $r \in [0,1]$. If f_r is a non-empty subset of N, then, by Proposition 2.7, $N \setminus f_r$ is closed under finite meet. If $f_r = \emptyset$, then, by Proposition 2.4, we are done.

(3) $\Rightarrow$ (1). Let $a, b \in N$, and $f(a \wedge b) = r \in [0,1]$. Since $a \wedge b \notin N \setminus f_r$, we can conclude from the statement (3) that $a \notin N \setminus f_r$ or $b \notin N \setminus f_r$. Hence $a \in f_r$ or $b \in f_r$, and which follows that $f(a \wedge b) \leq max\{f(a), f(b)\}$. The proof is now complete. $\square$

Proposition 3.10. *Let N be nexus over γ and f be an arbitrary fuzzy subnexus.*

(1) *If N is a chain, then f is a prime fuzzy subnexus.*
(2) *If f is a prime fuzzy subnexus and one to one, then N is a chain.*

Proof. (1) Suppose that $a, b \in N$, and $a \leq b$. Since $f(a) \geq f(b)$ so $f(a \wedge b) = f(a) = max\{f(a), f(b)\}$.

(2) Let $a, b \in N$ and $a \wedge b = c$. If $a \neq c$ and $b \neq c$, then since $c < a$, $c < b$ and f is one to one, we can conclude that $f(c) > f(a)$, and $f(c) > f(b)$. Therefore, $f(c) > max\{f(a), f(b)\} \geq f(a \wedge b)$, which is a contradiction. $\square$

Proposition 3.11. *Let $F : M \to N$ be a homomorphism between nexus. Then the following assertions hold:*

(1) *If g is a fuzzy subnexus of M, then $f = gF$ is a fuzzy subnexus of N.*
(2) *If g is a prime fuzzy subnexus of M, then $f = gF$ is a prime fuzzy subnexus of N.*

Proof. (1) It is clear that f is a fuzzy subset of N. Suppose that $a, b \in N$, and $a \leq b$. Since F is a homomorphism, we can conclude that $F(a) \leq F(b)$, which follows that $g(F(a)) \geq g(F(b))$. Hence, f is a fuzzy subnexus of N.

(2) For every $a, b \in N$,

$$\begin{aligned} f(a \wedge b) &= gF(a \wedge b) \\ &= g(F(a \wedge b)) \\ &= g(F(a) \wedge F(b)) \\ &= \leq \max\{g(F(a), g(F(b))\}. \end{aligned}$$

Hence, f is a prime fuzzy subnexus of N. □

Remark 3.12. Let $x \in N$ and $t \in (0,1]$. Then $< x^t >: N \to [0,1]$, defined by

$$< x^t > (a) = \begin{cases} t & x \in \uparrow a \\ \\ 0 & x \notin \uparrow a \end{cases}$$

is a fuzzy subnexus.

Remark 3.13. It is clear that if N is a nexus, and $|N| \leq 4$, then the nexus N is lineary ordered.

Proposition 3.14. *Let N be a nexus over γ. The following assertions are equivalent:*

(1) *Nexus N is lineary ordered.*
(2) *Every fuzzy subnexus of N is prime.*

Proof. (1) $\Rightarrow$ (2). Let $f \in Fsub(N)$, and $a, b \in N$. Hence, $a \leq b$ or $b \leq a$, say $a \leq b$, since nexus N is lineary ordered. Therefore, $f(a \wedge b) = f(a) \geq f(b)$, which follows that $f(a \wedge b) = max\{f(a), f(b)\}$.

(2) $\Rightarrow$ (1). Suppose that every fuzzy subnexus of N is prime, and $a, b \in N$. Put $a \wedge b = c$, and let $a \neq c$, $b \neq c$ and $t = \frac{1}{2} \in [0,1]$. It is clearly $t =< c^t > (c) \leq max\{< c^t > (a), < c^t > (b)\} = 0$, according to statement (2). This is a contradiction. Therefore, nexus N is lineary ordered. □

Proposition 3.15. *Let N be a nexus over γ, $a, b \in N$, and $r, t \in (0,1]$. Then the following assertions hold:*

(1) $< a^r > \wedge < b^t >=< (a \wedge b)^{r \wedge t} >$.
(2) $< (a \wedge b)^t > \wedge < a^t >=< (a \wedge b)^t >$.
(3) $< (a \vee b)^t > \wedge < a^t >=< a^t >$.

Proof. For every $x \in N$, $a, b \in \uparrow x$, if and only if $a \wedge b \in \uparrow x$. Hence, $< a^r > \wedge < b^t >=< (a \wedge b)^{r \wedge t} >$. The rest is similar. □

Proposition 3.16. *Let N be a nexus over γ, $a, b \in N$, and $r, t \in (0,1]$. We define $g : N \to [0,1]$ by*

$$g(x) = \begin{cases} r & a \in \uparrow x \& b \notin \uparrow x \\ s & a \notin \uparrow x \& b \in \uparrow x \\ r \vee s & a, b \in \uparrow x \\ 0 & a \notin \uparrow x \& b \notin \uparrow x \end{cases}$$

Then the following assertions hold:

(1) $g \in Fsub(N)$ *and* $g =< a^r > \vee < b^t >$.
(2) $< a^r > \vee < b^t > \leq < (a \vee b)^{r \vee s} >$.
(3) $< (a \wedge b)^t > \vee < a^t > = < a^t >$.
(4) $< (a \vee b)^t > \vee < a^t > = < (a \vee b)^t >$.
(5) $< a^r > \vee < a^t > = < a^{r \vee t} >$.

Proof. Evident. □

Proposition 3.17. *Let N be a nexus over γ, $a, b \in N$, and $r, t \in (0,1]$. The following assertions hold:*

(1) $a \leq b$, *if and only if* $< a^t > \leq < b^t >$.
(2) $r \leq t$, *if and only if* $< a^r > \leq < a^t >$.
(3) $< a^r > \wedge < a^t > = < a^{r \wedge t} >$.

Proof. (1) Let $a \leq b$. Since $a \in \uparrow x$, implies that $b \in \uparrow x$, we can conclude that $< a^t > (x) = t$ implies that $< b^t > (x) = t$. Hence, $< a^t > \leq < b^t >$.

Conversely, let $< a^t > \leq < b^t >$. Hence, $t =< a^t > (a) \leq < b^t > (a) \leq t$, i.e. $< b^t > (a) = t$. Therefore, $b \in \uparrow a$.

The rest is similar. □

Example 3.18. Let $\gamma = 3$, $N = \{(), (1), (2)\}$, and $h, f, g : N \to [0,1]$ be functions such that

$$f = \begin{pmatrix} () & (1) & (2) \\ 0.3 & 0.2 & 0.125 \end{pmatrix},$$

$$g = \begin{pmatrix} () & (1) & (2) \\ 0.4 & 0.35 & 0.1 \end{pmatrix}$$

and

$$h = \begin{pmatrix} () & (1) & (2) \\ 0.3 & 0.2 & 0.1 \end{pmatrix}.$$

It is clear that $h \in Fsub(N)$ is prime, and $f, g \in Fsub(N)$. Also, $f \wedge g \subseteq h$ but $f \nsubseteq h$ and $g \nsubseteq h$.

Proposition 3.19. *Let N be a nexus, and $f \in Fsub(N)$.*

(1) *If $|Imf| \leq 2$ and $\emptyset \neq f_* \in Psub(N)$, then $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, for every $g, h \in Fsub(N)$.*
(2) *If $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, for every $g, h \in Fsub(N)$, then $|Imf| \leq 2$.*
(3) *If $|Imf| = 2$ and for every $g, h \in Fsub(N)$, $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, then $\emptyset \neq f_* \in Psub(N)$.*

Proof. (1) If $|Imf| = 1$, then $Imf = \{1\}$, which finishes the proof.

Now, we assume that $|Imf| = 2$, then $Imf = \{t, 1\}$ with $t < 1$. Suppose that there exist two fuzzy subnexuses h and g over N, such that $g \wedge h \subseteq f$ but $g \not\subseteq f$ and $h \not\subseteq f$. Hence, there exist $x, y \in N$, such that $h(x) > f(x)$ and $g(y) > f(y)$. Since f_* is a prime subnexus, and $x, y \notin f_*$, we can conclude that $x \wedge y \notin f_*$, which follows that

$$(h \wedge g)(x \wedge y) = h(x \wedge y) \wedge g(x \wedge y) \geq h(x) \wedge g(y) > t = f(x \wedge y).$$

Thus $h \wedge g \not\subseteq f$, which is a contradicition. Thus $g \subseteq f$ or $h \subseteq f$.

(2) Let $|Imf| \geq 3$. Then there exists $a, b, c \in N$, such that $f(a) < f(b) < f(c)$. Now, we assume that $r, s \in (0, 1)$, such that $f(a) < r < f(b) < s < f(c)$. If $a \wedge b \in\uparrow x$, then, by Proposition 3.15,

$$(< a^r > \wedge < b^s >)(x) =< (a \wedge b)^{r \wedge s} > (x) = r < f(b) \leq f(a \wedge b) \leq f(x).$$

Therefore, $< a^r > \wedge < b^s > \subseteq f$, which follows that $< a^r > \subseteq f$ or $< b^s > \subseteq f$. If $< a^r > \subseteq f$, then $< a^r > (a) = r \leq f(a)$, which is a contradiction. Also, if $< b^s > \subseteq f$, then $< b^s > (b) = s \leq f(b)$, which is a contradiction. Hence, $|Imf| \leq 2$.

(3) Suppose that $f_* = \emptyset$. Then there exists $a, b \in N$, such that $f(a) = r < f(b) = s < 1$ and $Imf = \{r, s\}$. Now, we assume that $t, k \in (0, 1)$, such that $r < t < s < k < 1$. If $a \wedge b \in\uparrow x$, then, by Proposition 3.15,

$$(< a^t > \wedge < b^k >)(x) =< (a \wedge b)^{t \wedge k} > (x) = t < f(b) \leq f(a \wedge b) \leq f(x).$$

Therefore, $< a^t > \wedge < b^k > \subseteq f$, which follows that $< a^t > \subseteq f$ or $< b^k > \subseteq f$. Hence, $< a^t > (a) = t \leq f(a)$ or $< b^k > (b) = k \leq f(b)$, which is a contradiction. Thus $f_* \neq \emptyset$ and $f_* \neq N$. Let $a, b \in N$ such that $a \wedge b \in f_*$, $a \notin f_*$ and $b \notin f_*$. Then there exists $r \in (0, 1)$ such that $f(a) = f(b) < r < 1 = f(a \wedge b)$. If $a \wedge b \in\uparrow x$, then, by Proposition 3.15,

$$(< a^r > \wedge < b^r >)(x) =< (a \wedge b)^r > (x) = r < 1 = f(x).$$

Therefore, $< a^r > \wedge < b^s > \subseteq f$, which follows that $< a^r > \subseteq f$ or $< b^s > \subseteq f$. Hence, $< a^r > (a) = r \leq f(a)$ or $< b^r > (b) = r \leq f(b)$, which is a contradiction. Therefore, $f_* \in Psub(N)$. □

4. Fraction induced by nexus and fuzzy subnexus

In this section, the fractions of a nexus N over an ordinal is defined, and denoted by $S^{-1}N$, where S is a meet closed subset of $Fsub(N)$. It is shown that this structure is a meet-semilattice and isomorphic with $\{h\}^{-1}N$, where $h = \bigwedge S$. Also we show that every ideal of $S^{-1}N$ is of the form of $S^{-1}I$, where I is a subnexus of N.

Definition 4.1. A *meet closed subset* of $Fsub(N)$ is a non-empty subset S of $Fsub(N)$, such that $f \wedge g \in S$, for every $f, g \in S$.

Let S be a meet closed subset of $Fsub(N)$. Define the relation $\sim_S$ on $N \times S$ as follows:
$(a, f) \sim_S (b, g) \Leftrightarrow \exists h \in S \, \forall t \in (0, 1](< a^t > \wedge g \wedge h =< b^t > \wedge f \wedge h)$.
We will proved that $\sim_S$ is an equivalence relation. Let $a, b, c \in N$, $f, g, h \in S$, $(a, f) \sim_S (b, g)$, and $(b, g) \sim_S (c, h)$. Then there exists $h_1, h_2 \in S$ such that

$$< a^t > \wedge g \wedge h_1 =< b^t > \wedge f \wedge h_1$$

and

$$< b^t > \wedge h \wedge h_2 =< c^t > \wedge g \wedge h_2,$$

for every $t \in (0, 1]$. If $k = h_1 \wedge h_2 \wedge g$, then $k \in S$, and for every $t \in (0, 1]$, we have

$$\begin{aligned} < a^t > \wedge h \wedge k &= < a^t > \wedge h \wedge h_1 \wedge h_2 \wedge g \\ &= < a^t > \wedge g \wedge h_1 \wedge h_2 \wedge h \\ &= < b^t > \wedge f \wedge h_1 \wedge h_2 \wedge h \\ &= < b^t > \wedge h \wedge h_2 \wedge f \wedge h_1 \\ &= < c^t > \wedge g \wedge h_2 \wedge f \wedge h_1 \\ &= < c^t > \wedge f \wedge h_1 \wedge h_2 \wedge g \\ &= < c^t > \wedge f \wedge k \end{aligned}$$

Therefore, $\sim_S$ on $N \times S$ is transtive. It is clear that $\sim_S$ on $N \times S$ is reflexive and symmetric. Hence, the relation $\sim_S$ on $N \times S$ is an equivalence relation. Write $\frac{a}{f}$ for the class of (a, f). The set of all equivalence classes of $\sim_S$ on $N \times S$ is denoted by $S^{-1}N$, and it is called the fraction of N with respect to S.

Definition 4.2. Let S be a meet closed subset of $Fsub(N)$, and $\frac{a}{f}, \frac{b}{g} \in S^{-1}N$. Then we say $\frac{a}{f} \leq \frac{b}{g}$, if there exists $h \in S$ such that

$$< a^t > \wedge < b^t > \wedge f \wedge h = f \wedge g \wedge < a^t > \wedge h,$$

for every $t \in (0, 1]$.

Proposition 4.3. *Let S be a meet closed subset of $Fsub(N)$. Then $(S^{-1}N, \leq)$ is a meet-semilattice.*

Proof. It is clear that $\leq$ on $S^{-1}N$ is reflexive. Now, let $\frac{a}{f} \leq \frac{b}{g}$, $\frac{b}{g} \leq \frac{a}{f}$. Thus there exists $h_1, h_2 \in S$, such that:

$$< a^t > \wedge < b^t > \wedge f \wedge h_1 = f \wedge g \wedge < a^t > \wedge h_1$$

and

$$< b^t > \wedge < a^t > \wedge g \wedge h_2 = g \wedge f \wedge < b^t > \wedge h_2.$$

By the commutativity of $\wedge$, we have

$$\begin{aligned}(< a^t > \wedge g) \wedge (f \wedge g \wedge h_1 \wedge h_2) &= < a^t > \wedge < b^t > \wedge f \wedge h_1 \wedge g \wedge h_2 \\ &= g \wedge f \wedge < b^t > \wedge h_2 \wedge f \wedge h_1 \\ &= (< b^t > \wedge f) \wedge (f \wedge g \wedge h_1 \wedge h_2).\end{aligned}$$

Since S is a meet closed subset of N, we can conclude that $f \wedge g \wedge h_1 \wedge h_2 \in S$, which follows that $(a, f) \sim_S (b, g)$, and $\frac{a}{f} = \frac{b}{g}$. Thus $\leq$ on $S^{-1}N$ is antisymmetric.

Let $\frac{a}{f} \leq \frac{b}{g}$ and $\frac{b}{g} \leq \frac{c}{h}$, for some $\frac{a}{f}, \frac{b}{g}, \frac{c}{h} \in S^{-1}N$. Then there exists $h_1, h_2 \in S$, such that

$$< a^t > \wedge < b^t > \wedge f \wedge h_1 = f \wedge g \wedge < a^t > \wedge h_1$$

and

$$< b^t > \wedge < c^t > \wedge g \wedge h_2 = g \wedge h \wedge < b^t > \wedge h_2.$$

Hence,

$$\begin{aligned}(f \wedge h \wedge < a^t >) \wedge (g \wedge h_1 \wedge h_2) &= (f \wedge g \wedge < a^t > \wedge h_1) \wedge \\ &\quad (h \wedge h_2 \wedge g) \\ &= (f \wedge < a^t > \wedge < b^t > \wedge h_1) \wedge \\ &\quad (h \wedge h_2 \wedge g) \\ &= (g \wedge h \wedge < b^t > \wedge h_2) \wedge \\ &\quad (< a^t > \wedge f \wedge h_1) \\ &= (g \wedge < c^t > \wedge < b^t > \wedge h_2) \wedge \\ &\quad (< a^t > \wedge f \wedge h_1) \\ &= (f \wedge < b^t > \wedge < a^t > \wedge h_1) \wedge \\ &\quad (g \wedge < c^t > \wedge h_2) \\ &= (f \wedge g \wedge < a^t > \wedge h_1) \wedge \\ &\quad (g \wedge < c^t > \wedge h_2) \\ &= (f \wedge < c^t > \wedge < a^t >) \wedge \\ &\quad (g \wedge h_1 \wedge h_2).\end{aligned}$$

Since S is a meet closed subset of $Fsub(N)$, we can conclude that $g \wedge h_1 \wedge h_2 \in S$, which followes that $\frac{a}{f} \leq \frac{c}{h}$. Thus $\leq$ on $S^{-1}N$ is transitive, and $(S^{-1}N, \leq)$ is a partially ordered set.

Let $\frac{a}{f}, \frac{b}{g} \in S^{-1}N$. Since for every $t \in [0, 1]$, by Lemma 3.15,

$$(f \wedge g) \wedge < a^t > \wedge < (a \wedge b)^t > = (f \wedge g) \wedge f \wedge < (a \wedge b)^t >$$

and

$$(f \wedge g) \wedge < b^t > \wedge < (a \wedge b)^t > = (f \wedge g) \wedge g \wedge < (a \wedge b)^t >,$$

we can conclude that $\frac{a \wedge b}{f \wedge g} \leq \frac{a}{f}$ and $\frac{a \wedge b}{f \wedge g} \leq \frac{b}{g}$. Now, let $\frac{c}{h} \in S^{-1}N$, such that $\frac{c}{h} \leq \frac{a}{f}$ and $\frac{c}{h} \leq \frac{b}{g}$. Then there exists $v, w \in S$, such that

$$h \wedge < a^t > \wedge < c^t > \wedge v = h \wedge f \wedge < c^t > \wedge v,$$

and

$$h \wedge < b^t > \wedge < c^t > \wedge w = h \wedge g \wedge < c^t > \wedge w.$$

Hence,

$$\begin{aligned}(h \wedge f \wedge g \wedge < c^t >) \wedge (v \wedge w) &= (h \wedge f \wedge < c^t > \wedge v) \wedge \\ &\quad (h \wedge g \wedge < c^t > \wedge w) \\ &= (h \wedge < a^t > \wedge < c^t > \wedge v) \wedge \\ &\quad (h \wedge < b^t > \wedge < c^t > \wedge w) \\ &= (h \wedge < a^t > \wedge < b^t > \wedge < c^t >) \wedge \\ &\quad (v \wedge w) \\ &= (h \wedge < (a \wedge b)^t > \wedge < c^t >) \wedge \\ &\quad (v \wedge w).\end{aligned}$$

Since S is a meet closed subset of N, we can conclude that $v \wedge w \in S$, which follows that $\frac{c}{h} \leq \frac{a \wedge b}{f \wedge g}$. Therefore, $\frac{a}{f} \wedge \frac{b}{g} = \frac{a \wedge b}{f \wedge g}$. □

Proposition 4.4. *Let S be a meet closed subset of $Fsub(N)$. For every $a \in N$ and $f, g \in S$, $\frac{a}{f} = \frac{a}{g}$ in $S^{-1}N$.*

Proof. Since $(< a^t > \wedge g) \wedge (f \wedge g) = (< a^t > \wedge f) \wedge (f \wedge g)$, and $f \wedge g \in S$, we have $(a, f) \sim_S (a, g)$, and $\frac{a}{f} = \frac{a}{g}$ in $S^{-1}N$. □

Proposition 4.5. *Let N be a nexus over γ, and let S be a meet closed subset of $Fsub(N)$.*

1. *Every ideal of $S^{-1}N$ is of the form of $S^{-1}I$, where I is a subnexus of N.*
2. *If K is a finite ideal of $S^{-1}N$, and $h = \bigwedge S \in S$, then there exists a cyclic subnexus I of N such that $K = S^{-1}I$.*
3. *If M is a prime ideal of $S^{-1}N$, then there exists $I \in Psub(N)$ such that $M = S^{-1}I$.*
4. *If M is a maximal ideal of $S^{-1}N$, then there exists $I \in Sub(N)$ such that $M = S^{-1}I$, and I is a maximal subnexus of N.*

Proof. (1) Let K be an ideal of $S^{-1}N$, and

$$I = \{a \in N | \tfrac{a}{f} \in K \text{ for some } f \in S\}.$$

Suppose that $a, b \in N$, $b \in I$, and $a \leq b$. Then there exists $f \in S$, such that $\frac{b}{f} \in K$. By Proposition 3.17, $< a^t > \leq < b^t >$ for every $t \in (0, 1]$. Then

$$< a^t > \wedge < b^t > \wedge f = < a^t > \wedge f$$

for every $t \in (0, 1]$. Hence, $\frac{a}{f} \leq \frac{b}{f} \in K$. Since K is an ideal of $S^{-1}N$, we can conclude that $\frac{a}{f} \in K$, which follows that $a \in I$. Now, by Proposition 2.3, I is a subnexus of N, and it is clear that $K = S^{-1}I$.

(2) Let K be a finite ideal of $S^{-1}N$. It is well known that every finite directed subset of $S^{-1}N$ has the largest element. Since K is

a directed lower set, we can conclude that there exists $\frac{a}{f} \in K$, such that $K = \downarrow \frac{a}{f}$. We put $I = \downarrow a$, and we claim that $K = S^{-1}I$. Let $\frac{b}{g} \in K$. Then there exists $k \in S$, such that, for every $t \in (0,1]$, $g \wedge < a^t > \wedge < b^t > \wedge k = g \wedge f \wedge < b^t > \wedge k$, which follows that

$$\begin{aligned}(< (a \wedge b)^t > \wedge g) \wedge (g \wedge f \wedge k) &= (< a^t > \wedge < b^t > \wedge g) \wedge \\ &\quad (g \wedge f \wedge k) \\ &= (< b^t > \wedge f) \wedge (g \wedge f \wedge k),\end{aligned}$$

for every $t \in (0,1]$. Therefore, $\frac{b}{g} = \frac{a \wedge b}{f} \in S^{-1}I$. Now, let $b \in I$ and $g \in S$. Then, by Proposition 3.17,

$$\begin{aligned}g \wedge < a^t > \wedge < b^t > \wedge h &= < b^t > \wedge h \\ &= g \wedge f \wedge < b^t > \wedge h,\end{aligned}$$

for every $t \in (0,1]$. Hence, $\frac{b}{g} \leq \frac{a}{f} \in K$. Since K is an ideal of $S^{-1}N$, we can conclude that $\frac{b}{g} \in K$. The proof is now complete.

(3) Let $I = \{a \in N | \frac{a}{f} \in M \text{ for some } f \in S\}$. Then, by statement (1), $M = S^{-1}I$. Let $a, b \in N$, such that $a \wedge b \in I$. Then $\frac{a \wedge b}{f} \in S^{-1}I$ for some $f \in S$. Since $\frac{a \wedge b}{f} = \frac{a}{f} \wedge \frac{b}{f}$ and $S^{-1}I$ is a prime ideal, we can conclude that $\frac{a}{f} \in S^{-1}I$ or $\frac{b}{f} \in S^{-1}I$. Hence, $a \in I$ or $b \in I$, i.e. $I \in Psub(N)$.

(4) Let $I = \{a \in N | \frac{a}{f} \in M \text{ for some } f \in S\}$. Then, by statement (1), $M = S^{-1}I$. Suppose I is not a maximal subnexus of N. Then there exist a subnexus J between I and N. Put $M_1 = S^{-1}J$. Then M_1 is an ideal of $S^{-1}N$, and $S^{-1}I \subset S^{-1}J$, which is contradicition. □

Lemma 4.6. *Let S be a meet closed subset of $Fsub(N)$, and $h = \bigwedge S$. For every $a, b \in N$ and $f, g \in S$*

1. *If $(a,h) \sim_S (b,h)$, then $(a,h) \sim_{\{h\}} (b,h)$.*
2. *If $h \in S$ and $(a,h) \sim_{\{h\}} (b,h)$, then $(a,h) \sim_S (b,h)$.*
3. *If $\frac{a}{f} \leq \frac{b}{g}$ in $S^{-1}N$, then $\frac{a}{h} \leq \frac{b}{h}$ in $\{h\}^{-1}N$.*

Proof. (1) We first suppose that $(a,h) \sim_S (b,h)$. Then there exists $v \in S$ such that

$$< a^t > \wedge h =< a^t > \wedge h \wedge v =< b^t > \wedge h \wedge v =< b^t > \wedge h.$$

It follows that $(a,h) \sim_{\{h\}} (b,h)$.

(2) By hypothesis, $< a^t > \wedge h =< b^t > \wedge h$. Since $h \in S$, we can conclude that $(a,h) \sim_S (b,h)$.

(3) Since $\frac{a}{f} \leq \frac{b}{g}$ in $S^{-1}N$, we can conclude that there exists $v \in S$, such that

$$< a^t > \wedge < b^t > \wedge f \wedge v = f \wedge g \wedge < a^t > \wedge v.$$

It is clear that $f \wedge v \wedge h = h = f \wedge g \wedge v \wedge h$. Then:

$$\begin{aligned} h \wedge < a^t > \wedge < b^t > &= < a^t > \wedge < b^t > \wedge f \wedge v \wedge h \\ &= f \wedge g \wedge < a^t > \wedge h \wedge v \\ &= h \wedge < a^t >, \end{aligned}$$

i.e. $\frac{a}{h} \leq \frac{b}{h}$ in $\{h\}^{-1}N$. □

Proposition 4.7. *Let S be a meet closed subset of $Fsub(N)$, and $h = \bigwedge S$. We define $\varphi : S^{-1}N \longrightarrow \{h\}^{-1}N$ with $\varphi(\frac{a}{f}) = \frac{a}{h}$. Then we have the following conclusions:*

(1) *φ is an onto meet-semilattice homomorphism.*
(2) *If $h \in S$, then φ is one to one. In particular, this shows if $h \in S$, then $S^{-1}N \cong \{h\}^{-1}N$ as meet-semilattices.*

Proof. (1) By Lemma 4.6, φ is well defined, and it also preserves the order. Let $\frac{a}{f}, \frac{b}{g} \in S^{-1}N$. Then, by the proof of Proposition 4.3,

$$\varphi(\frac{a}{f} \wedge \frac{b}{g}) = \varphi(\frac{a \wedge b}{f \wedge g}) = \frac{a \wedge b}{h} = \frac{a}{h} \wedge \frac{b}{h} = \varphi(\frac{a}{f}) \wedge \varphi(\frac{b}{g}).$$

Therefore, φ is an onto meet-semilattice homomorphism.

(2) Let $\frac{a}{f}, \frac{b}{g} \in S^{-1}N$, and $\varphi(\frac{a}{f}) = \varphi(\frac{b}{g})$. Then $\frac{a}{h} = \frac{b}{h}$ and

$$< a^t > \wedge h \wedge g =< a^t > \wedge h =< b^t > \wedge h =< b^t > \wedge f \wedge h,$$

for every $t \in (0, 1]$. Since $h \in S$, we can conclude that $\frac{a}{f} = \frac{b}{g}$, which followes that φ is one to one. □

Proposition 4.8. *Let N be a nexus over γ, and let S be a meet closed subset of $Fsub(N)$. If $h = \bigwedge S$, then $\{h\}^{-1}N \cong \overbrace{\downarrow h}$ as meet-semilattices, where $\overbrace{\downarrow h} = \{h \wedge < a^1 >; a \in N\}$.*

Proof. We define $\varphi : \{h\}^{-1}N \longrightarrow \overbrace{\downarrow h}$ with $\varphi(\frac{a}{h}) =< a^1 > \wedge h$. For every $a, b \in N$,

$$\frac{a}{h} = \frac{b}{h} \Rightarrow < a^1 > \wedge h =< b^1 > \wedge h \Rightarrow \varphi(\frac{a}{h}) = \varphi(\frac{b}{h}).$$

Hence, φ is well defined. It is clear that φ is onto. Now, let $\frac{a}{h} \neq \frac{b}{h}$. We show that $\varphi(\frac{a}{h}) \neq \varphi(\frac{b}{h})$. Since $\frac{a}{h} \neq \frac{b}{h}$, there exists $t \in (0, 1]$, such that $< a^t > \wedge h \neq < b^t > \wedge h$. If $t = 1$, then $\varphi(\frac{a}{h}) \neq \varphi(\frac{b}{h})$. Let $t < 1$ and $< a^1 > \wedge h =< b^1 > \wedge h$. For every $x \in N$,

(1) If $a, b \in \uparrow x$, then $< a^t > (x) = t =< b^t > (x)$, which follows that $(< a^t > \wedge h)(x) = t \wedge h(x) = (< b^t > \wedge h)(x)$.

(2) If $a, b \notin \uparrow x$, then $< a^t > (x) = 0 =< b^t > (x)$, which follows that $(< a^t > \wedge h)(x) = 0 = (< b^t > \wedge h)(x)$.

(3) If $a \in \uparrow x$ and $b \notin \uparrow x$, then

$$\begin{aligned} h(x) &= 1 \wedge h(x) \\ &= (< a^1 > \wedge h)(x) \\ &= (< b^1 > \wedge h)(x) \\ &= 0 \wedge h(x) \\ &= 0, \end{aligned}$$

which follows that $(< a^t > \wedge h)(x) = 0 = (< b^t > \wedge h)(x)$.

(4) Similarly, if $a \notin \uparrow x$ and $b \in \uparrow x$, then

$$(< a^t > \wedge h)(x) = 0 = (< b^t > \wedge h)(x)$$

Therefore, $< a^t > \wedge h =< b^t > \wedge h$, which is a contradicition. Then $< a^1 > \wedge h \neq < b^1 > \wedge h$. Hence φ is one to one. Let $\frac{a}{h}, \frac{b}{h} \in \{h\}^{-1}N$. Then, by Proposition 3.15 and the proof of Proposition 4.3,

$$\begin{aligned} \varphi(\tfrac{a}{h} \wedge \tfrac{b}{h}) &= \varphi(\tfrac{a \wedge b}{h}) \\ &= < (a \wedge b)^1 > \wedge h \\ &= (< a^1 > \wedge h) \wedge (< b^1 > \wedge h) \\ &= \varphi(\tfrac{a}{h}) \wedge \varphi(\tfrac{b}{h}). \end{aligned}$$

Therefore, φ is a meet-semilattice isomorphism. □

Corollary 4.9. *Let N be a nexus over γ, and let S_1, S_2 be meet closed subsets of $Fsub(N)$. If $\bigwedge S_1 = \bigwedge S_2 \in S_1 \cap S_2$, then $S_1^{-1}N \cong S_2^{-1}N$ as meet-semilattices.*

Proof. By Propositions 4.7 and 4.8, it is clear. □

Proposition 4.10. *Let N be a nexus over γ, and $\{()\} \neq X \subseteq N \setminus \{()\}$ be closed under finite meet. Then for every $t \in (0, 1]$, $S_t = \{< a^t > |a \in X\}$ is closed under finite meet, and there exists $b \in X$, such that $< b^t >= \bigwedge S_t$.*

Proof. By Proposition 3.15, S_t is closed under finite meet. Since $X \subseteq N$, and X is closed under finite meet, we can conclude from Corollary 2.8 that there exists $b \in X$, such that $b = \bigwedge X$. By Proposition 3.17, $< b^t >= \bigwedge_{a \in X} < a^t >$. □

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FUZZY NEXUS OVER AN ORDINAL

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علی اکبر استاجی، تکتم حقدادی و جواد فرخی
دانشگاه حکیم سبزواری، دانشگاه صنعتی بیرجند

ما در این مقاله زیر پیوندهای فازی از یک پیوند N را تعریف می‌کنیم. هم چنین به مطالعه زیرپیوندهای فازی اول و زیرپیوندهای خارج قسمتی می‌پردازیم. در نهایت نشان می‌دهیم که اگر S یک زیر مجموعه بسته مقطعی از زیرپیوندهای فازی N باشد و $S \in S$ ،$h = \bigwedge S$، آن گاه $\{h\}^{-1}N$ و $S^{-1}N$ به عنوان نیم مشبکه های مقطعی یکریخت خواهند بود.

کلمات کلیدی: پیوند، عدد ترتیبی ، زیرپیوندهای فازی اول و پیوند خارج قسمتی .