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AMENABILITY OF VECTOR-VALUED GROUP ALGEBRAS

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Abstract. The purpose of carrying out this work is to develop the amenability notations for vector-valued group algebras. We prove that $L^1(G, A)$ is approximately weakly amenable where A is a unital separable Banach algebra. We give the necessary, and sufficient conditions for the existence of a left invariant mean on $L^\infty(G, A^*)$, $LUC(G, A^*)$, $WAP(G, A^*)$, and $C_0(G, A^*)$.

1. Introduction

In 1972, B. E. Johnson proved that a locally compact group G is amenable if and only if $L^1(G)$ is amenable [7]. The concept of Johnson's amenability for Banach algebras has been a main stream in the theory of Banach algebras. Here we develop the concept of Johnson's amenability for vector-valued Banach algebras.

Let G be a locally compact group with a fixed left Haar measure m and A be a unital separable Banach algebra. Let $L^1(G, A)$ be the set of all measurable vector-valued (equivalence classes of) functions $f : G \to A$ such that $\|f\|_1 = \int_G \|f(t)\|dm(t) < \infty$. Equipped with the norm $\|.\|_1$ and the convolution product * specified by:

$$f * g(x) = \int f(t)g(t^{-1}x)dm(t) \ (f, g \in L^1(G, A)),$$

$L^1(G, A)$ is a Banach algebra. We prove the analogues of the classical results on amenability of Banach algebras. We show that G is amenable

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if and only if $L^1(G, A)$ is amenable for each unital separable Banach algebra A. The symbol $M(G, A)$ stands for the space of regular A-valued Borel measures of bounded variation on G. The space $L^1(G, A)$ is a closed two-sided ideal of $M(G, A)$.

Let $L^\infty(G, A^*)$ be the set of all functions f of G into A^* that are scalarwise measurable and $N_\infty(\|f\|) = \text{loc ess sup}_{t\in G}(\|f(t)\|) < \infty$. From now on, A will be a separable Banach algebra. By Theorem 8.18.2 in [5], the dual of $L^1(G, A)$ may be identified with $L^\infty(G, A^*)$. Note that the dual of $L^1(G, A)$ is in general not $L^\infty(G, A^*)$. We show that every continuous derivation, from $L^1(G, A)$ into $L^\infty(G, A^*)$ is approximately inner, that is, of the form

$$D(a) = \lim_\alpha (F_\alpha.a - a.F_\alpha)$$

for some $\{F_\alpha\}_{\alpha\in I} \in L^\infty(G, A^*)$.

Let $C(G, A^*)$ be the space of bounded continuous functions from G into A^*, let $C_0(G, A^*)$ be the continuous functions from G into A^* vanishing at infinity and let $C_{00}(G, A^*)$ be the continuous functions from G into A^* with compact support under the norm $\|f\| = \sup_{t\in G} \|f(t)\|$. For $f \in L^\infty(G, A^*)$, set $L_x f(t) = f(xt)(x,\ t \in G)$. Then f is called left uniformly continuous, if the map $x \mapsto L_x f$ from G into $L^\infty(G, A^*)$ is continuous with respect to $N_\infty(\|f\|)$ on $L^\infty(G, A^*)$. The set of uniformly continuous functions is denoted by $LUC(G, A^*)$. A function $f \in C(G, A^*)$ is called weakly almost periodic if the set $\{L_x f :\ x \in G\}$ is relatively compact in the weak-topology on $C(G, A^*)$. The set of weakly almost periodic functions are denoted by $WAP(G, A^*)$. In the case $A = \mathbb{C}$, the complex field, these spaces are denoted by $L^1(G)$, $M(G)$, $C(G)$, $C_0(G)$, $C_{00}(G)$, $LUC(G)$ and $WAP(G)$. For general terms in vector-valued functions, we follow [5].

The left invariant means on spaces of vector-valued functions were first considered by Dixmier in [3]. In this work, we set up a relation between a vector-valued mean and a scalar-valued mean, by which we will be able to translate many important results developed in the classic theory. We also present some of the properties of left invariant means on $LUC(G, A^*)$, $WAP(G, A^*)$ and $C_0(G, A^*)$. Our references for the vector-valued integration theory are [1], [2] and [5].

2. Main results

Definition 2.1. Let A be a Banach algebra and X be a subspace of $L^\infty(G, A^*)$. A map $M : L^\infty(G, A^*) \to A^*$ is called a mean on X if
(i) M is linear;
(ii) For each $f \in X$, $M(f)$ belongs to the set $\overline{co\{f(x) :\ x \in G\}}$, where

closure is taken in the weak*-topology, and coX denotes the convex hull of a set X.

If X is also left invariant, then M is called left invariant if $M(L_a f) = M(f)$ for each $a \in G$ and $f \in X$. Dixmier showed in [3] that if m is a left invariant mean on $L^\infty(G)$. Then m induces a left invariant mean M on $L^\infty(G, A^*)$ such that $\langle M(f), a\rangle = m(\langle f(.), a\rangle)$ for each $a \in A$, here $\langle f(.), a\rangle$ denotes the functions $x \mapsto \langle f(x), a\rangle$.

Let A and B be two Banach algebras, and let A be a closed ideal in B. For each $a \in A$, put $\rho_a(b) := \|ba\| + \|ab\|$ $(b \in B)$, then ρ_a is a seminorm. The topology defined on B by these seminorms is called the strict topology. We write B^{*st} for the dual of B with respect to the strict topology.

Theorem 2.2. *Let G be a locally compact group. Then G is amenable if and only if $L^1(G, A)$ is amenable for each unital separable Banach algebra A.*

Proof. Suppose that $L^1(G, A)$ is amenable for each unital separable Banach algebra A. Consider $A = \mathbb{C}$. Then G is amenable [7]. Conversely, let X be a Banach $L^1(G, A)$-bimodule, and let $D : L^1(G, A) \to X^*$ be a continuous derivation. By Proposition 8.1 in [4], $L^1(G, A)$ has a bounded approximate identity. Then there is no loss of generality if we suppose that X is pseudo-unital. By Proposition 2.1.6 in [12], there is a unique $\overline{D} : M(G, A) \to X^*$ that extends D and is continuous with respect to the strict topology on $M(G, A)$, and the weak*-topology on X^*.

We can embed G into $M(G, A)$. The map $\delta_x : G \to M(G, A)$ given by $\delta_x(H) = \chi_H(x)e_A$ for each $x \in G$ and $H \subseteq G$ is the required embedding. We claim that $\Delta = \{\delta_x : \ x \in G\}$ is dense in $M(G, A)$ with respect to the strict topology. We assume to the contrary that μ is not in $\overline{\Delta}$ where closure is taken in the strict topology, thus there are some $f \in M(G, A)^{*st}$, such that $\langle f, \mu\rangle = 1$, $\langle f, \delta_x\rangle = 0$ for each $x \in G$. By Proposition 23.18 and Proposition 23.33 in [2], the map $x \mapsto \delta_x$ is continuous with respect to the strict topology. This implies that $\langle f, \mu\rangle = \int \langle f, \delta_x\rangle d\mu(x) = 0$ [11].

It suffices to show that $\overline{D}\mid_\Delta$ is inner. Because, if $\mu \in M(G, A)$, there will be a net $\{\delta_{x_\alpha}\}_{\alpha \in I}$ in Δ with $\delta_{x_\alpha} \to \mu$ in the strict topology. So $\overline{D}(\delta_{x_\alpha}) \to \overline{D}(\mu)$ in the weak*-topology. But, there is $\beta^* \in X^*$ such that $\overline{D}(\mu) = w_k^* - \lim_\alpha(\beta^*.\delta_{x_\alpha} - \delta_{x_\alpha}.\beta^*)$. Now let $\beta \in X$. Since X is pseudo-unital Banach $L^1(G, A)$-bimodule, then there are $f_1, \ f_2 \in L^1(G, A)$

and $\acute{\beta} \in X$ such that $\beta = f_1.\acute{\beta}.f_2$. Hence:

$$\begin{aligned}\langle \beta^*.\delta_{x_\alpha} - \delta_{x_\alpha}.\beta^*, \beta\rangle &= \langle \beta^*.\delta_{x_\alpha} - \delta_{x_\alpha}.\beta^*, f_1.\acute{\beta}.f_2\rangle \\ &\to \langle \beta^*.\mu - \mu.\beta^*, \beta\rangle.\end{aligned}$$

Consequently:

$$\overline{D}(\mu) = \beta^*.\mu - \mu.\beta^*.$$

Consider the constant function $1 \in L^\infty(G)$. There is a function $f_0 : G \to A^*$ by $f_0(x) = \langle 1, x\rangle a^*$ ($a^* \in A^*$, $x \in G$). More generally, each function $g \in L^\infty(G)$ has the form $\langle f(.), a\rangle$ with $a \neq 0$, $a^* \in A^*$, $\langle a^*, a\rangle = 1$ and $f(x) = \langle g, x\rangle a^*$. For each $\beta \in X$, define $\langle \Lambda_\beta, \mu\rangle = \langle \overline{D}(\mu)\mu, \beta\rangle$ ($\mu \in M(G, A)$). Put $\Lambda_{\acute{\beta}} = \Lambda_\beta|_\Delta$. Now define $\langle \rho_\beta, x\rangle = \langle \Lambda_{\acute{\beta}}, \delta_x\rangle = \langle \overline{D}(\delta_x).\delta_x, \beta\rangle$ ($x \in G$). Since $\rho_\beta \in L^\infty(G)$, then there are some $f_\beta \in L^\infty(G, A^*)$ such that $f_\beta(x) = \rho_\beta(x)a^*$ ($a^* \in A^*$, $x \in G$). By assumption, there is $m \in L^\infty(G)^*$ such that $\langle m, 1\rangle = 1$, $\langle m, L_x f\rangle = \langle m, f\rangle$ for each $f \in L^\infty(G)$ and $x \in G$. Thus m induces a left invariant mean M on $L^\infty(G, A^*)$. Now put $\langle \lambda, \beta\rangle = \langle M, f_\beta\rangle$. The space X becomes a Banach Δ-module via $\delta_x \Box \beta = \beta$, $\beta \Box \delta_x = \delta_x.\ \beta.\ \delta_x$ ($x \in G$, $\beta \in X$). Then dual actions on X^* are given by $\beta^* \Box \delta_x = \beta^*$, $\delta_x \Box \beta^* = \delta_x.\ \beta^*.\ \delta_x$.

Define $\overline{D_0}(\delta_x) = \overline{D}(\delta_x).\ \delta_x$. It is routinely checked that $\overline{D_0}$ is a derivation, and $\overline{D}$ is inner if and only if $\overline{D_0}$ is inner for this new module structure. It remains to be shown that $\overline{D_0}$ is inner. For $\delta_{x_0} \in M(G, A)$,

$$\begin{aligned}\langle \delta_{x_0}\Box\lambda - \lambda\Box\delta_{x_0}, \beta\rangle &= \langle \lambda, \beta\Box\delta_{x_0} - \delta_{x_0}\Box\beta\rangle \\ &= \langle \lambda, \beta\Box\delta_{x_0} - \beta\rangle = \langle M, f_{\beta\Box\delta_{x_0} - \beta}\rangle.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\langle f_{\beta\Box\delta_{x_0}-\beta}, x\rangle &= \langle \rho_{\beta\Box\delta_{x_0}-\beta}, x\rangle a^* = \overline{D}(\delta_x).\delta_x(\beta\Box\delta_{x_0} - \beta)a^* \\ &= \overline{D_0}(\delta_x)(\beta\Box\delta_{x_0} - \beta)a^* = \langle \delta_{x_0}\Box\overline{D_0}(\delta_x) - \overline{D_0}(\delta_x), \beta\rangle a^* \\ &= \langle \overline{D_0}(\delta_{x_0}\delta_x) - \overline{D_0}(\delta_{x_0})\Box\delta_x - \overline{D_0}(\delta_x), \beta\rangle a^* \\ &= \langle \overline{D_0}(\delta_{x_0}\delta_x), \beta\rangle a^* - \langle \overline{D_0}(\delta_{x_0}), \beta\rangle a^* - \rho_\beta(x)a^* \\ &= \langle \rho_\beta, x_0x\rangle a^* - \langle \overline{D_0}(\delta_{x_0}), \beta\rangle\langle 1, x\rangle a^* - \rho_\beta(x)a^* \\ &= f_\beta(x_0x) - \langle \overline{D_0}(\delta_{x_0}), \beta\rangle f_0(x) - f_\beta(x),\end{aligned}$$

then $\langle \delta_{x_0}\Box\lambda - \lambda\Box\delta_{x_0}, \beta\rangle = -\langle \overline{D_0}(\delta_{x_0}), \beta\rangle$. It follows that $\overline{D_0}(\delta_{x_0}) = \delta_{x_0}\Box\lambda_0 - \lambda_0\Box\delta_{x_0}$ where $\lambda_0 = -\lambda$. Thus $\overline{D_0}$ is inner. □

Theorem 2.3. *Let G be a locally compact group. Then the following statements are equivalent:*

(i) *G is amenable.*

(ii) *For every unital separable Banach algebra A, there exists a bounded net $\{\psi_\alpha\}_{\alpha\in I} \subseteq L^1(G,A)$ such that $\|\delta_x * \psi_\alpha - \psi_\alpha\|_1 \to 0$ whenever $x \in G$.*

(iii) *For every unital separable Banach algebra A, there exists a bounded net $\{\psi_\alpha\}_{\alpha\in I} \subseteq L^1(G,A)$ such that for every compact set $K \subseteq G$, $\|\psi * \psi_\alpha - \psi_\alpha\|_1 \to 0$ uniformly for all $\psi \in L^1(G,A)$ with $\int_{G\setminus K} \|\psi(t)\|dm(t) = 0$.*

Proof. Consider $A = \mathbb{C}$. Then Theorem 6.7 in [10] yields $(ii) \Longrightarrow (i)$, $(iii) \Longrightarrow (i)$.

$(i) \Longrightarrow (ii)$ Suppose that G is amenable. Let m be a left invariant mean on $L^\infty(G)$, and M be an induced mean on $L^\infty(G,A^*)$. Choose $a \in A$ such that, $\|a\| = 1$. Define $\langle \Gamma_a, f\rangle = \langle M(f), a\rangle$ for each $f \in L^\infty(G,A^*)$. Regards Γ_a as an element of $L^\infty(G,A^*)^*$. The rest of the proof is essentially the same as the Lemma 6.3 in [10].

$(i) \Longrightarrow (iii)$ This is just a re-statement of Proposition 6.7 in [10]. □

Let $\{e_\alpha\}_\alpha$ be a bounded approximate identity for $L^1(G)$ and e_A be an identity in A. By Proposition 8.1 in [4], $\{e_\alpha \otimes e_A\}_\alpha$ is a bounded approximate identity for $L^1(G)\hat{\otimes}A$ where $\hat{\otimes}$ denotes the completion of usual tensor product of Banach spaces with respect to the projective tensor norm. We consider $\{e_\alpha \otimes e_A\}_\alpha$ as an element in $(L^1(G)\hat{\otimes}A)^{**}$ and $F \in (L^1(G)\hat{\otimes}A)^*$. Using exactly the same notation as in [6], we put $\langle (e_\alpha \otimes e_A), F\rangle = \int F d(e_\alpha \otimes e_A)$. Given a dual Banach space X^* and $F \in B(L^1(G), A; X^*)$, we define $\int F d(e_\alpha \otimes e_A) \in X^*$ by

$$\langle \int F d(e_\alpha \otimes e_A), x\rangle = \int \langle F(f,a), x\rangle d(e_\alpha \otimes e_A)(f,a),$$

where $f \in L^1(G),\ a \in A$ *and* $x \in X$.

Theorem 2.4. *Let G be a locally compact group and let A be a unital separable Banach algebra. Then $L^1(G,A)$ is approximately weakly amenable.*

Proof. Let $D : L^1(G,A) \to L^1(G,A)^*$ be a continuous derivation. It is well-known that the space $L^1(G,A)$ is isometrically isomorphic to $L^1(G)\hat{\otimes}A$. Therefore we define $F : L^1(G) \times A \to L^1(G,A)^*$ by $F(f,a) = D(f \otimes a)$. Put, $g_\alpha = \int F(f,a)d(e_\alpha \otimes e_A)(f,a)$. We know that $L^1(G,A)^* \cong L^\infty(G,A^*)$. Then for each $F(f,a) \in L^1(G,A)^*$, its image under isometry onto $L^\infty(G,A^*)$ is a map whose values at $x \in G$ is $F(f,a)(x) = \langle D(f \otimes a), x\rangle$. We put $\overline{F}_x : L^1(G) \times A \to A^*$ with $\overline{F}_x(f,a) = F(f,a)(x)$, $f \in L^1(G)$, $a \in A$, and $x \in G$. Note that $\overline{F}_x \in B(L^1(G), A; A^*)$. By the above argument, we define

$\int \overline{F}_x(f,a)d(e_\alpha \otimes e_A)(f,a) \in A^*$ by

$$\langle \int \overline{F}_x(f,a)d(e_\alpha \otimes e_A)(f,a), c\rangle = \int \langle \overline{F}_x(f,a), c\rangle d(e_\alpha \otimes e_A)(f,a), (c \in A).$$

The map $x \mapsto g_\alpha(x) = \int F(f,a)(x)d(e_\alpha \otimes e_A)(f,a)$ is a scalarwise measurable function, and $N_\infty(\|g_\alpha(x)\|) < \infty$ for each α. Then, by Theorem 8.18.2 in [5], there is a map κ_{g_α} in $B(A, L^\infty(G))$ such that, $\langle \kappa_{g_\alpha}(a), f\rangle = \int f(x)\langle g_\alpha(x), a\rangle dm(x)$ for each $f \in L^1(G)$, and $a \in A$, where κ_{g_α} is defined by $\kappa_{g_\alpha}(a) = \langle g_\alpha(x), a\rangle$.

Using the same notation as in [6], we have $e_\alpha \otimes e_A = \int f \otimes g d(e_\alpha \otimes e_A)(f,g)$. Moreover, $e_\alpha \otimes e_A$ is a bounded approximate identity for $L^1(G)\hat{\otimes}A$. Therefore, for each $F : L^1(G) \times A \to L^1(G,A)^*$, $f,\ g \in L^1(G)$, and $a,\ b \in A$, we have

$$\begin{aligned}\lim_\alpha \int F(fg,ab)d(e_\alpha \otimes e_A)(f,a) &= \lim_\alpha \langle \int (fg \otimes ab)d(e_\alpha \otimes e_A)(f,a), F\rangle \\ &= \lim_\alpha \langle \int (gf \otimes ba)d(e_\alpha \otimes e_A)(f,a), F\rangle \\ &= \lim_\alpha \int F(gf,ba)d(e_\alpha \otimes e_A)(f,a).\end{aligned}$$

Hence

$$\begin{aligned}&\lim_\alpha (g \otimes b)\langle \acute{f}, \kappa_{g_\alpha}(\acute{a})\rangle \\ = \ &\lim_\alpha \int \acute{f}(x)\langle (g \otimes b).g_\alpha(x), \acute{a}\rangle dm(x) \\ = \ &\lim_\alpha \int \acute{f}(x)\langle \int (g \otimes b).D(f \otimes a)(x)d(e_\alpha \otimes e_A)(f,a), \acute{a}\rangle dm(x) \\ = \ &\lim_\alpha \int \acute{f}(x)\langle \int D(gf \otimes ba)(x)d(e_\alpha \otimes e_A)(f,a), \acute{a}\rangle dm(x) \\ - \ &\lim_\alpha \int \acute{f}(x)\langle \int D(g \otimes b).(f \otimes a)(x)d(e_\alpha \otimes e_A)(f,a), \acute{a}\rangle dm(x) \\ = \ &\int \acute{f}(x) \lim_\alpha \langle \int F(gf,ba)(x)d(e_\alpha \otimes e_A)(f,a), \acute{a}\rangle dm(x) \\ - \ &\int \acute{f}(x) \lim_\alpha \langle D(g \otimes b) \int (f \otimes a)(x)d(e_\alpha \otimes e_A)(f,a), \acute{a}\rangle dm(x) \\ = \ &\int \acute{f}(x) \lim_\alpha \langle \int F(fg,ab)(x)d(e_\alpha \otimes e_A)(f,a), \acute{a}\rangle dm(x) \\ - \ &\int \acute{f}(x)\langle D(g \otimes b)(x), \acute{a}\rangle dm(x)\end{aligned}$$

$$= \int \acute{f}(x) \lim_{\alpha} \langle \int F(f,a)(x) d(e_\alpha \otimes e_A)(f,a), \acute{a} \rangle dm(x)(g \otimes b)$$
$$- \int \acute{f}(x) \langle D(g \otimes b)(x), \acute{a} \rangle dm(x)$$
$$= \lim_{\alpha} \int \acute{f}(x) \langle g_\alpha(x), \acute{a} \rangle dm(x)(g \otimes b) - \int \acute{f}(x) \langle D(g \otimes b)(x), \acute{a} \rangle dm(x)$$
$$= \lim_{\alpha} \langle \acute{f}, \kappa_{g_\alpha}(\acute{a}) \rangle (g \otimes b) - \langle \acute{f}, \kappa_{D(g \otimes b)}(\acute{a}) \rangle$$

for all $g \otimes b \in L^1(G) \otimes A$, $\acute{a} \in A$ and $\acute{f} \in L^1(G, A)$. Consequently,

$$\lim_{\alpha} ((g \otimes b)\kappa_{g_\alpha}(\acute{a}) - \kappa_{g_\alpha}(\acute{a})(g \otimes b)) = -\kappa_{D(g \otimes b)}(\acute{a})$$
$$\lim_{\alpha} \langle (g \otimes b).g_\alpha(x), \acute{a} \rangle - \langle g_\alpha(x).(g \otimes b), \acute{a} \rangle = -\langle D(g \otimes b)(x), \acute{a} \rangle$$

for all $g \otimes b \in L^1(G) \otimes A$ and $\acute{a} \in A$. It follows that

$$D(g \otimes b) = \lim_{\alpha} ((g \otimes b).\acute{g_\alpha} - \acute{g_\alpha}.(g \otimes b))$$

for all $g \otimes b \in L^1(G) \otimes A$, where $\acute{g_\alpha} = -g_\alpha$. □

It is known that G is amenable if and only if $LUC(G)$ has a left invariant mean. It is interesting to have a direct proof of this fact. We present a vector version of this characterization.

Theorem 2.5. *Let G be a locally compact group, and let A be a unital separable Banach algebra. Then:*

(i) $L^\infty(G, A^*)L^1(G, A) = LUC(G, A^*)$.

(ii) *G is amenable if and only if $LUC(G, A^*)$ has a left invariant mean.*

Proof. (*i*) Let $f \in LUC(G, A^*)$. Let $\{U_\alpha\}$ be a net of neighborhoods of e directed downwards. Let $\{\nu_\alpha\}_{\alpha \in I}$ be an approximate identity of norm 1 in $L^1(G, A)$ such that $supp\nu_\alpha \subseteq U_\alpha$. Given $\epsilon > 0$, there exists $\alpha_0 \in I$ such that for each $\alpha \geq \alpha_0$ and $y \in U_\alpha$, $N_\infty(\|f_y - f\|) < \epsilon$. Then

$$\begin{aligned}
& |\langle f.\nu_\alpha, \mu \rangle - \langle f, \mu \rangle| \\
= \ & |\langle f, \nu_\alpha * \mu \rangle - \langle f, \mu \rangle| \\
= \ & \Big| \int \langle f(t), \nu_\alpha * \mu(t) \rangle dm(t) - \int \langle f(t), \mu(t) \rangle dm(t) \Big| \\
= \ & \Big| \int \langle f(t), \int \mu(y^{-1}t) d\nu_\alpha(y) \rangle dm(t) - \int \langle f(t), \mu(t) \rangle dm(t) \Big| \\
= \ & \Big| \int \langle f(yt), \int \mu(t) d\nu_\alpha(y) \rangle dm(t) - \int \langle f(t), \mu(t) \rangle dm(t) \Big| \\
\leq \ & N_\infty(\|f_y - f\|) \|\mu\| \|\nu_\alpha\|_1.
\end{aligned}$$

On the other hand, $f.\nu_\alpha \in L^\infty(G,A^*)L^1(G,A)$, and $L^\infty(G,A^*)L^1(G,A)$ is a Banach space. It follows that $f \in L^\infty(G,A^*)L^1(G,A)$. It is easy to see that $L^\infty(G,A^*)\, L^1(G,A) \subseteq LUC(G,A^*)$, and so $L^\infty(G,A^*)L^1(G,A)$ $= LUC(G,A^*)$.

(ii) This follows from (i) and the proof of Lemma 3.2 in [9]. □

Let A be a Banach algebra. Recall that a functional $f \in A^*$ for which $\{f.a :\ \|a\| \le 1,\ a \in A\}$ is relatively compact in the weak-topology of A^* is said to be weakly almost periodic. The set of weakly almost periodic functionals on A is denoted by $WAP(A)$ (see [8]). It is known that $WAP(G) = WAP(L^1(G))$ [13]. Note that for $f \in L^\infty(G,A^*)$ and $\mu \in M(G,A)$, we define $\langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle$ for every $\nu \in L^1(G,A)$.

Theorem 2.6. *Let G be a locally compact group, and A be a unital separable Banach algebra. Then:*

- (i) *If $f \in L^\infty(G,A^*)$, then $f \in WAP(L^1(G,A))$ if and only if $\{f\delta_x :\ x \in G\}$ is relatively weakly compact in $L^\infty(G,A^*)$.*
- (ii) *$WAP(L^1(G,A)) = WAP(G,A^*)$.*
- (iii) *$WAP(L^1(G,A))$ has a left invariant mean.*

Proof. (i) Let $f \in WAP(L^1(G,A))$. It is known that $C_0(G,A)^* = M(G,A^*)$ [2]. Now consider $\delta_x \in M(G,A^*)$. By the Hhan Banach Theorem, we may assume that $m \in L^\infty(G,A^*)^*$ is an extension of δ_x with norm one. Then there is a net $\{\mu_\alpha\}_{\alpha\in I}$ in $L^1(G,A)$ with $\|\mu_\alpha\| \le 1$ such that $\mu_\alpha \to m$ in the weak*-topology. Hence, for every $\phi \in L^1(G,A)$,

$$\langle \phi.f, \mu_\alpha\rangle \to \langle \phi.f, m\rangle.$$

Since $\{f.\mu :\ \|\mu\| \le 1, \mu \in L^1(G,A)\}$ is relatively weakly compact, then there is an element $g \in L^\infty(G,A^*)$ and a subnet $\{\mu_\beta\}_{\beta\in I}$ of $\{\mu_\alpha\}_{\alpha\in I}$ such that $f.\mu_\beta \to g$ in the weak-topology. On the other hand, $\langle m, \phi.f\rangle = \langle \delta_x, \phi.f\rangle = \langle f\delta_x, \phi\rangle$, and so $g = f\delta_x$. Thus the set $\{f\delta_x :\ x \in G\}$ is contained in $\overline{\{f.\mu :\ \|\mu\| \le 1, \mu \in L^1(G,A)\}}$, where closure is taken in the weak-topology, and the compactness of $\overline{\{f\delta_x :\ x \in G\}}$ follows from the compactness of $\overline{\{f.\mu :\ \|\mu\| \le 1\}}$.

Conversely, suppose that $f \in L^\infty(G,A^*)$ and $\mu \in L^1(G,A)$ with $\|\mu\| \le 1$, and $\{f\delta_x :\ x \in G\}$ is relatively weakly compact. By the Krein-Smulian Theorem, $co\{f\delta_x :\ x \in G\}$ is relatively weakly compact. We claim that

$$f\mu \in \overline{co\{f\delta_x :\ x \in G\}},$$

where closure is taken in the weak-topology. We assume to the contrary that $f\mu$ is not in $\overline{co\{f\delta_x :\ x \in G\}}$. By the Hahn Banach Theorem, there exists $F \in L^\infty(G, A^*)^*$, such that

$$Re|\langle F, f\mu\rangle| \geq \gamma_1 > \gamma_2 > Re|\langle F, f\delta_x\rangle|,$$

where $\gamma_1,\ \gamma_2 \in \mathbb{R}$, and $x \in G$. By Theorem 8.14.8 in [5], the integral $\int f\delta_x d\mu$ belongs to $L^\infty(G, A^*)$. Moreover, for any $\nu \in L^1(G, A)$, $\langle \int f\delta_x d\mu, \nu\rangle = \int \langle f.\delta_x, \nu\rangle d\mu = \int \langle f, \delta_x * \nu\rangle d\mu = \langle f, \mu * \nu\rangle = \langle f\mu, \nu\rangle$ (see Chapter 3 in [11]). By Theorem 3.28 in [11], we have

$$|\langle F, f\mu\rangle| = |\int \langle F, f\delta_x\rangle d\mu| \leq \int |\langle F, f\delta_x\rangle| d|\mu|(x) < Re\langle F, f\mu\rangle.$$

This is a contradiction, and so $\{f.\mu : \|\mu\| \leq 1, \mu \in L^1(G, A)\}$ is contained in the closure of $co\{f\delta_x : \ x \in G\}$, and the compactness of $\overline{\{f.\mu : \|\mu\| \leq 1, \mu \in L^1(G, A)\}}$ follows from the compactness of $\overline{co\{f\delta_x : \ x \in G\}}$. Consequently, $f \in WAP(L^1(G, A))$, and the proof is complete.

(ii) Let $f \in WAP(G, A^*)$. Then the set $\{L_x f : x \in G\}$ is relatively weakly compact in $C(G, A^*)$. Note that for each $x \in G$, we have $\langle f\delta_x, \mu\rangle = \langle f, \delta_x * \mu\rangle = \int \langle f(t), \int \mu(y^{-1}t) d\delta_x(t)\rangle dm(t) = \langle L_x f, \mu\rangle$. Then, by the Eberlien-Smulian Theorem, $\{f\delta_x : \ x \in G\}$ is relatively weakly compact in $L^\infty(G, A^*)$, and from (i), $f \in WAP(L^1(G, A))$.

Conversely, let $f \in WAP(L^1(G, A))$. The map $x \mapsto f\delta_x$ is continuous with respect to the weak*-topology. From (i), $\{f\delta_x : x \in G\}$ is relatively weakly compact. Then $x \to f\delta_x$ is continuous with respect to the weak-topology. Now, let $\{U_\alpha\}$ be a net of neighborhood of e directed downwards. Let $\{\nu_\alpha\}_{\alpha \in I}$ be an approximate identity of norm 1 in $L^1(G, A)$ such that $supp\nu_\alpha \subseteq U_\alpha$. Given $\epsilon > 0$ and $F \in L^\infty(G, A^*)^*$, there exists an α_0 such that for each $\alpha \geq \alpha_0$ and $x \in U_\alpha$, $|\langle F, f\delta_x\rangle - \langle F, f\rangle| < \epsilon$, and so $|\int \langle F, f\delta_x\rangle d\nu_\alpha - \langle F, f\rangle| < \epsilon$. By Theorem 8.14.8 in [5] and the Krein-Smulian Theorem, $\int f\delta_x d\nu_\alpha \in L^\infty(G, A^*)$. Moreover, for any $\mu \in L^1(G, A)$, $\langle \int f\delta_x d\nu_\alpha(x), \mu\rangle = \langle f\nu_\alpha, \mu\rangle$. Therefore, for each $\alpha \geq \alpha_0$,

$$\begin{aligned} |\langle F, f\nu_\alpha\rangle - \langle F, f\rangle| &= |\langle F, \int f\delta_x d\nu_\alpha(x)\rangle - \langle F, f\rangle| \\ &= |\int \langle F, f\delta_x\rangle d\nu_\alpha - \langle F, f\rangle| < \epsilon. \end{aligned}$$

So $f\nu_\alpha \to f$ in the weak-topology. An argument similar to that in the proof of Lemma 6.3 in [10] shows that we can find a bounded net $\{\nu_\alpha\}_{\alpha \in I}$ consisting of convex combination of elements in $\{\nu_\alpha\}_{\alpha \in I}$ such that $f\nu_\alpha \to f$ in the norm topology. In addition, $L^\infty(G, A^*)L^1(G, A)$ is a Banach space. Then from Theorem 2.5, $f \in C(G, A^*)$. Since

$\{f\delta_x : x \in G\}$ is relatively weakly compact, then by the Eberlian-Smulian Theorem, $\{L_x f : x \in G\}$ is relatively weakly compact in $C(G, A^*)$. Hence, $f \in WAP(G, A^*)$.

(iii) It is an immediate consequence of Theorem 4.3 in [14] and (ii). □

In the next theorem, we present an interesting property about a left invariant mean when considered on $C_0(G, A^*)$. Analogous to the scalar function case, we can easily obtain the following theorem.

Theorem 2.7. *Let G be a non-compact amenable group and let $f \in C_0(G, A^*)$. If M is left invariant mean on $L^\infty(G, A^*)$, then $|M(f)| = 0$.*

Proof. Let M be a left invariant mean on $L^\infty(G, A^*)$ and $f \in L^\infty(G, A^*)$. Then the set $\{f(t) : \|f(t)\| \leq \|f\|_\infty\}$ is weak*-closed in A^* and $M(f) \in \overline{\{f(x) : x \in G\}}$, where closure is taken in the weak*-topology. It follows that $|M(f)| \leq \|f\|_\infty$. Using the Urysohn Lemma, it is easy to see that $\overline{C_{00}(G, A^*)}^{\|\cdot\|} = C_0(G, A^*)$. Then it is enough to prove that the result in the case where $f \in C_{00}(G, A^*)$. Let $K = suppf$. There exists an infinite sequence $\{a_n\}_{n\in\mathbb{N}}$ in G such that $a_0 = e$, and $(a_i K) \bigcap (a_j K) = \emptyset$, whenever $i,\ j \in \mathbb{N}, i \neq j$ [10]. Put $g_n = \Sigma_{i=0}^n L_{a_i} f\ (n \in \mathbb{N})$. For any $n \in \mathbb{N}$,

$$|M(g_n)| = |nM(f)| \leq \|g_n\|_\infty = \|f\|_\infty.$$

Consequently, $M(f) = 0$. □

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AMENABILITY OF VECTOR VALUED GROUP ALGEBRAS

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هدف این مقاله توسیع مفهوم میانگین‌پذیری جبرهای گروهی برداری‌مقدار است. نشان می‌دهیم اگر A جبر باناخ جدایی پذیر یکدار باشد، آنگاه جبر گروهی $L^1(G, A)$ میانگین‌پذیر ضعیف تقریبی است. همچنین شرایط لازم و کافی برای وجود میانگین پایای چپ روی $LUC(G, A^*), WAP(G, A^*)$ و $C_0(G, A^*)$ را بررسی می‌کنیم.

کلمات کلیدی: میانگین‌پذیری، جبر باناخ، مشتق، جبر گروهی، میانگین پایا.