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IDEALS IN EL-SEMIHYPERGROUPS ASSOCIATED TO ORDERED SEMIGROUPS

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Abstract. In this work, we attempt to investigate the connection between various types of ideals (for examples (m,n)-ideal, bi-ideal, interior-ideal, quasi-ideal, prime-ideal and maximal-ideal) of an ordered semigroup $(S,\cdot,\leq)$ and the corresponding, hyperideals of its EL-hyperstructure $(S,*)$ (if exists). Moreover, we construct the class of EL-Γ-semihypergroup, associated to a partially-ordered Γ-semigroup.

1. Introduction

The application of mathematics in other branches of science plays a vital role and they represent, in the recent decades, one of the purposes of the study of the expert of hyperstructure theory all over the world. The hyperstructure theory was first introduced in 1934 by the French mathematician Marty [11]. He, at the 8th Congress of Scandinavian Mathematicians, defined hypergroups as a natural generalization of groups based on the notion of hyperoperation. Since then, a number of different hyperstructures have been widely studied by many mathematicians. A recent book on hyperstructures [4] has pointed out their applications to fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

EL-hyperstructures, which was first introduced by Chvalina in [3], are hypercompositional structures constructed, from a partially/quasi

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(semi)group using a construction known as Ending lemma or Ends lemma. Lots of papers regarding this topic have been written like Hoskova [9, 10], Novak [12, 13, 14, 15, 16, 17], Rackova [18, 19], and Rosenberg [20] and others. Among them, Novak in [14] studied sub-hyperstructures of EL-hyperstructures and in [12], he discussed some interesting results of important elements, in this family of hyperstructures. Then, in [13], Novak studied some basic properties of EL-hyperstructures like invertibility, normality, property of being closed, and ultra closed, regularity and etc. Also Rackova gave a description of subhyprgroups of EL-hypergroups derived from a quasi-ordered group [18]. In addition, after the Ends lemma extension by Rackova in [19](Theorem 4), there is a natural question that "Is it possible to go further to stronger hyperstructure-like canonical hypergroups, strongly canonical hypergroups and etc". A positive answer to this question would mean that various ring-like EL-hyperstructures could be studied extensively. Using the Ends lemma construction, Novak in [15], considered the potential of the Ends lemma to create ring-like hyperstructures and in [17], constructed the n-ary hyperstructures from binary quasi-ordered semigroups. Even though EL-hyperstructures have been widely used and studied, the ideals of EL-hyperstructures have not been studied and investigated yet. This work aims at studying the various kinds of ideals in a quasi-ordered semigroup and its EL-(semi)hypergroup. More precisely, we wish to see which property, being bi-ideal, interior-ideal, quasi-ideal and etc, of a given ideal of an ordered semigroup can be herited by hyperideals of the associated EL-hyperstructure.

2. Preliminaries

In this part, we recall some basic definitions and properties that we will consider later.

Definition 2.1. A hypergroupoid or a multigroupoid is a pair $(H, \circ)$, where H is a non-empty set, and $\circ : H \times H \longrightarrow \wp^*(H)$ is a binary hyperoperation also called a multioperation. ($\wp^*(H)$ is the system of all non-empty subsets of H.) A semihypergroup is an associative hypergrupoid, i.e. a hypergrupoid satisfying the equality $a \circ (b \circ c) = (a \circ b) \circ c$ for every triad $a, b, c \in H$. If moreover, the semihypergroup H satisfies $a \circ H = H = H \circ a$, for all $a \in H$, it is called a *hypergroup*. A non-empty subset $G \subseteq H$ is called a *subhypergroup* of $(H, \circ)$, if $a \circ G = G = G \circ a$ for all $a \in H$.

In the above definition, if A an B are two non-empty subsets of H and $x \in H$, then $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ and $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$.

For a deeper insight into the basic hyperstructure theory (cf [4] and [5]).

Since the theory of ordered structures is dealt with ordered relations, we need to recall some definitions in this respect.

Definition 2.2. Binary relation $\leq$ is called *quasi-order* if it is reflexive and transitive. Also, if the binary relation $\leq$ is reflexive, transitive and anti-symmetric, then it is known as a partially-order relation.

Definition 2.3. By a partially/quasi-ordered (semi)group, we mean a triple $(S, \cdot, \leq)$, where $(S, \cdot)$ is a (semi)group, and $\leq$ is a partially/quasi order relation on S such that for all $x, y, z \in S$ with the property $x \leq y$, there holds $(x \cdot z) \leq (y \cdot z)$ and $(z \cdot x) \leq (z \cdot y)$. The element $0 \in S$ is called zero element if $0 \leq x$, and $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$.

Moreover, the notation $[x)_{\leq}$, used below stands for the set $\{s \in S; x \leq s\}$ and also $[A)_{\leq} = \bigcup_{x \in A} [x)_{\leq}$. Similarly, $(x]_{\leq} = \{s \in S; s \leq x\}$ and $(A]_{\leq} = \bigcup_{x \in A} (x]_{\leq}$. A non-empty subset A of S is called an upper end of S if for all $a \in A$, there holds $[a)_{\leq} \subseteq A$. If there exists an element $a \in A$ such that there exists $x \in S \setminus A$ such that $x \in [a)_{\leq}$, we say that A is not an upper end of S because of the element x.

Definition 2.4. Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup, and $\varnothing \neq I \subseteq S$. Then,

a) I is called a left (resp. right) ideal of S if $S \cdot I \subseteq I$ $(I \cdot S \subseteq I)$.

b) I is called a left (resp. right) ordered ideal of S if it is a left (resp. right) ideal of S, and in addition, $a \in I$ and $b \leq a$ imply that $b \in I$(i.e. $(I]_{\leq} = I$).

c) I is called an (ordered) ideal of S if it is both a left and right (ordered) ideal of S.

d) I is called a (an ordered) bi-ideal of S if $I \cdot S \cdot I \subseteq I$ (and $(I]_{\leq} = I$). A bi-ideal I of S is called a (an ordered) subidempotent of S if $I \cdot I \subseteq I$ (and $(I]_{\leq} = I$).

e) I is called an (ordered) interior-ideal of S if $S \cdot I \cdot S \subseteq I$ (and $(I]_{\leq} = I$).

f) I is called a (an ordered) (m, n)-ideal of S if $I^m \cdot S \cdot I^n \subseteq I$ (and $(I]_{\leq} = I$).

g) I is called a (an ordered) quasi-ideal of S if $S \cdot I \bigcap I \cdot S \subseteq I$ (and $(I]_{\leq} = I$).

h) An ideal P of semigroup S is called prime if $I \cdot J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. P is called an ordered prime ideal of S if it is prime and $(P]_{\leq} = P$.

i) An ideal P of semigroup S is called semiprime if $I \cdot I \subseteq P$ implies

that $I \subseteq P$ for all ideal $I \subseteq S$ or equivalently $x^2 \in P$ implies that $x \in P$ for all $x \in S$. P is called an ordered semiprime ideal of S if it is semiprime and $(P]_{\leq} = P$.
j) The semigroup S is called simple if it has no proper ideal i.e. $aS = Sa = S$ for all $a \in S$.
k) The semigroup S is called 0-simple if it has zero element, and, in addition, 0 is the only proper ideal of S.
l) The semigroup S is called an *idempotent semigroup* or a *band* if $E(S) = S$, in which $E(S)$ is the set of all idempotent of S (i.e. $x = x^2$ for all $x \in S$).

The EL–hyperstructures or *Ends lemma* based hyperstructures are hyperstructures constructed from quasi/partially-ordered (semi)groups using "Ends lemma". This concept was first introduced by Chvalina in 1995 [3]. In particular, Chvalina proved that:

Lemma 2.5. *([3], Theorem 1.3) Let $(S, \cdot, \leq)$ be a partially-ordered semigroup. Binary hyperoperation $\circ : S \times S \longrightarrow \wp^*(S)$, defined by $a \circ b = [a \cdot b)_{\leq} = \{x \in S, a \cdot b \leq x\}$, is associative. The semihypergroup $(S, \circ)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.*

Theorem 2.6. *([3], Theorem 1.4) Let $(S, \cdot, \leq)$ be a partially-ordered semigroup. The following conditions are equivalent:*
I) For any pair $(a, b) \in S^2$, there exists a pair $(c, c_1) \in S^2$ such that $b \cdot c \leq a$ and $c_1 \cdot b \leq a$.
II) The associated semihypergroup $(S, \circ)$ is a hypergroup.

The following Theorem, which was proved by Rackova in her Ph.D thesis, extends the Ends lemma. The proof can also be found in [18].

Theorem 2.7. *([18], Theorem 4) Let $(S, \cdot, \leq)$ be a quasi-ordered group, and $(S, *)$ be the associated EL-hyperstructure. Then, $(S, *)$ is a transposition hypergroup.*

3. (Hyper)Ideals

In this section, we introduce different types of hyperideals, and then try to discover the relation between the ideals of an ordered semigroup and the hyperideals of its associated EL-hyperstructure.

Definition 3.1. Let $(H, \circ)$ be a semihypergroup. A non-empty subset I of H is called a left (resp. right) *hyperideal* of H if $H \circ I \subseteq I$ $(I \circ H \subseteq I)$. A non-empty subset I of H is called a hyperideal of H if it is both a left and right hyperideal of H.

Example 3.2. Let $S = [0, 1]$. Then, S with the hyperoperation $x \circ y = [1, xy]$ is a semihypergroup. Suppose $I = [0, t]$ for $t \in [0, 1]$. It is easy to see that I is a hyperideal of semihypergroup $(S, \circ)$.

Theorem 3.3. *Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup, $(S, *)$ be its associated EL-hyperstructure, and I be a left (ordered) ideal of S, which is also an upper end of S. Then I is a left hyperideal of $(S, *)$. A similar statement holds for right (ordered) ideals.*

Proof. Suppose I is a left (ordered) ideal of $(S, \cdot, \leq)$. We have $S * I = \bigcup_{x \in S, y \in I} x * y = \bigcup_{x \in S, y \in I} [x \cdot y)_{\leq}$. Now for all $(x, y) \in S \times I$ there holds $x \cdot y \in I$ since I is a left ideal of S. Also, $[x \cdot y)_{\leq} \subseteq I$ because I is an upper end of S. Therefore, $S * I \subseteq I$. □

Example 3.4. Let S = {a,b,c,d,f} be an ordered semigroup with the following multiplication table:

·	a	b	c	d	f
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
f	f	f	f	f	f .

We define the order relation $\leq$ as follows:

$$\leq := \{(a, c), (f, b), (f, d), (a, a), (b, b), (c, c), (d, d), (f, f)\}.$$

By a simple computation, we can obtain the semihypergroup $(S, *)$ with using the following table:

*	a	b	c	d	f
a	{a,c}	b	c	b	b
b	b	b	b	b	b
c	{a,c}	b	c	b	b
d	d	b	d	b	b
f	{b,d,f}	{b,d,f}	{b,d,f}	{b,d, f}	{b,d,f}

Now consider $I = \{f, b, d\}$ and $J = \{b, c, d, f\}$. It is easy to see that I is an ordered right ideal, and J is an ordinary (not ordered) left ideal of $(S, \cdot, \leq)$. Also both of them are the upper ends of S. A simple computation shows that $I * S \subseteq I$ and $S * J \subseteq J$, which means that I is a right hyperideal and J is a left hyperideal of $(S, *)$.

Corollary 3.5. *Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup, $(S, *)$ be its associated EL-hyperstructure, and I be an (ordered) ideal of S, which is also an upper end of S. Then, I is a hyperideal of $(S, *)$.*

Remark 3.6. In Theorem 3.3, the condition of being an upper end of S is an essential condition. Consider the following example.

Example 3.7. Let $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, and $S = M_2(\mathbb{N}_0)$. Then S with a usual matrix multiplication is a semigroup. Define the relation $\leq$ on S as follows: for all $A, B \in S$ and $i, j \in \{1, 2\}$, $A \leq B \Longleftrightarrow a_{ij} \leq b_{ij}$. It is easy to check that $(S, \cdot, \leq)$ is a partially-ordered semigroup. In addition, let $I = \{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} | a, b \in \mathbb{N}_0\} \subseteq S$. I is a left-ordered ideal of $(S, \cdot, \leq)$ (i.e. $SI \subseteq I$), which is not an upper end of S. We claim that I can not be a left hyperideal of $(S, *)$. Indeed, there holds:

$$S * I = \bigcup_{X \in S, Y \in I} X * Y = \bigcup_{x,y,z,t,a,b \in \mathbb{N}_0} [\begin{bmatrix} xa + by & 0 \\ az + bt & 0 \end{bmatrix})_{\leq} = S.$$

Thus $S * I = S \nsubseteq I$.

Remark 3.8. It is easy to see that a left (resp. right) hyperideal of $(S, *)$ is always an ordinary left (resp. right) ideal of $(S, \cdot, \leq)$ (We shall prove later) but it is not necessarily an upper end of S. In the next theorem, we show that if the quasi-ordered semigroup $(S, \cdot, \leq)$ possesses an identity element, i.e. it becomes a quasi-ordered monoid, then the converse of Theorem 3.3 would hold for ordinary ideals.

Theorem 3.9. *Let $(S, \cdot, \leq)$ be a quasi-ordered monoid, and $(S, *)$ be its associated EL-hyperstructure. Then I is a left (reps. right) ideal of S, which is also an upper end of S if and only if I is a left (reps. right) hyperideal of $(S, *)$.*

Proof. In order to prove $\Longleftarrow$, we have to show that:
1) $S \cdot I \subseteq I$ $(I \cdot S \subseteq I)$;
2) the subset I of S is an upper end of S.

Let $x \in S$, and $y \in I$ be arbitrary elements. Since the relation $\leq$ is reflexive, there holds $x \cdot y \in [x \cdot y)_{\leq} = x * y \subseteq S * I \subseteq I$ which implies that $S \cdot I \subseteq I$. A similar verification shows that $I \cdot S \subseteq I$.

Now suppose (by contradiction) that I is not an upper end of S. Thus there exists $x \in I$ such that $[x)_{\leq} \nsubseteq I$. Now for the identity element $e \in S$ there holds $e * x = [e \cdot x)_{\leq} = [x)_{\leq} \nsubseteq I$, which implies that $S * I \nsubseteq I$, a contradiction.

□

Example 3.10. Let $S = (\mathbb{N} - \{1\}, +, \leq)$ with usual sum and ordering of numbers. It is clear that S has no identity element. Also let $I = \{6, 8, 9, 10, 11, 12, \cdots\} \subset S$. Then using the Ends lemma construction, we can see that in the associated EL-hyperstructure $(S, \oplus)$, there holds $I \oplus S = S \oplus I = \{8, 10, 11, 12, \cdots\} \subset I$, which means that I is a hyperideal of $(S, \oplus)$. In addition, $I + S \subseteq I \oplus S \subset I$ and $S + I \subseteq S \oplus I \subset I$. This implies that I is an ideal of $(S, +, \leq)$. But, I is not an upper end of S because of element 7, Since $7 \in [6)_{\leq}$ but $7 \notin I$.

Definition 3.11. ([6], p.177) Let $(H, \circ)$ be a semihypergroup. Then $(H, \circ)$ is called simple if it has no proper hyperideal (i.e. $H \circ x \circ H = H$ for all $x \in H$).

Theorem 3.12. *Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup in which $(S, \cdot)$ is simple. Then the associated EL-semihypergroup $(H, \circ)$ is simple.*

Proof. Let I be an arbitrary hyperideal of $(S, \circ)$. As mentioned in Remark 3.8, I can be regarded as an ideal of semigroup, $(S, \cdot)$ which implies that $I = S$ due to simplicity of $(S, \cdot)$. □

Remark 3.13. The converse of Theorem 3.12 is not true, i.e. there is non-simple ordered semigroup $(S, \cdot, \leq)$ such that its associated EL-semihypergroup $(H, \circ)$ is simple. Look at the following example:

Example 3.14. ([2], Table 11) Let S = {a,b,c} be an ordered semigroup with the following multiplication table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	c	a

We define the order relation $\leq$ as follows:

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (c, c)\}.$$

Consider $I = \{a\}$, for which, there holds $a \cdot S = S \cdot a = a$. Thus $(S, \cdot)$ is not simple. By a simple computation, we can achieve the semihypergroup $(S, *)$ using the following table:

$*$	a	b	c
a	S	S	S
b	S	{b}	S
c	S	{c}	S

It can be easily checked that $(S,*)$ has no proper hyperidead. Notice that $x*S = S*x = S$ for all $x \in S$.

Definition 3.15. ([6], p.177) Let $(H,*)$ be a semihypergroup. An element $0 \in H$ is called *left zero scalar element* (resp. *right zero scalar element*) if for all $x \in H$, there holds $0*x = 0$ (resp. $x*0 = 0$). If 0 is both left and right zero scalar element, then 0 is called *zero scalar element*.

Definition 3.16. ([6], p.177) A semihypergroup $(H,*)$ with zero scalar element is called *0-simple* if $H*x*H = H$ for all $x \in H - \{0\}$. (It has no proper nontrivial ideal).

Theorem 3.17. *Let $(S,\cdot,\leq)$, be a non-trivial quasi-ordered semigroup with zero element. Then,*
1) The associated EL-semihypergroup $(S,*)$ is a hypergroup. i.e. $(S,*)$ is a simple semihypergroup.
2) $(S,*)$ does not have zero scalar element, i.e. $(S,*)$ is not 0-simple semihypergroup.

Proof. 1) There exists an element $0 \in S$ such that $0 \leq x$, and $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$. We have, $0*x = [0\cdot x)_{\leq} = [0)_{\leq} = S$ and $0*S = \bigcup\limits_{x\in S} 0*x = S$. Now, for an arbitrary element $s \in S$, there holds $s*S = s*(0*S) = (s*0)*S = S*S = S$. Similarly, we can see that $S = S*s$ for all $s \in S$.
2) Suppose that $x \in S$ is the zero scalar element of $(S,*)$. Thus $x*s = x = s*x$ for all $s \in S$. Now, for the zero element $0 \in (S,\cdot,\leq)$, we have $0 = x*0 = [x\cdot 0)_{\leq} = [0)_{\leq} = S$, which is impossible. □

Definition 3.18. Let $(H,\circ)$ be a semihypergroup. A non-empty subset I of H is called a left (resp. right) *maximal hyperideal* of H if there is no proper left (resp. right) hyperideal J such that $I \subset J \subset S$.

Theorem 3.19. *Let $(S,\cdot,\leq)$ be a quasi-ordered monoid, and $(S,*)$ be its associated EL-hyperstructure. Then, I is maximal among all left (reps. right) ideals of S, which are also an upper end of S if and only if I is maximal among all left (reps. right) hyperideals of $(S,*)$.*

Proof. We prove just the necessity part. The proof of sufficiency part is similar. Suppose J is a left (resp. right) hyperideal of $(S,*)$ such that $I \subset J \subseteq S$. Then, J is a left (resp. right) ideal of $(S,\cdot,\leq)$, which is also an upper end of S by Theorem 3.9. Now, the maximality of I implies that $J = S$.

□

Example 3.20. Let $I = \{f, b, d\}$ in Example 3.4. It can be easily be checked that I is a maximal right ideal of $(S, \cdot, \leq)$. Now, by Theorem 3.19, it is a maximal right hyperideal of $(S, *)$.

Theorem 3.21. *Let $(S, \cdot, \leq)$ be a quasi-ordered monoid, and $(S, *)$ be its associated EL-hyperstructure. Then, I is minimal among all left (reps. right) ideals of S, which are also an upper end of S if and only if I is minimal among all left (reps. right) hyperideals of $(S, *)$.*

Proof. The proof is similar to the proof of Theorem 3.19. □

Remark 3.22. Notice that the necessity parts of Theorem 3.9, Theorem 3.19 and Theorem 3.21 are not true for the ordered ideals of $(S, \cdot, \leq)$. Actually, for the ordered ideals of $(S, \cdot, \leq)$, there holds:

Theorem 3.23. *Let $(S, \cdot, \leq)$ be a quasi-ordered monoid, and $(S, *)$ be its associated EL-hyperstructure. Then I is a left (reps. right) ordered ideal of S, which is also an upper end of S if and only if I is a left (reps. right) hyperideal of $(S, *)$ with the property $(I]_{\leq} = I$.*

Theorem 3.24. *Let $(S, \cdot, \leq)$ be a quasi-ordered monoid, and $(S, *)$ be its associated EL-hyperstructure. Then I is maximal among all left (reps. right) ordered ideals of S, which are also an upper end of S if and only if I is maximal among all left (reps. right) hyperideals of $(S, *)$ with the property $(I]_{\leq} = I$.*

Theorem 3.25. *Let $(S, \cdot, \leq)$ be a quasi-ordered monoid, and $(S, *)$ be its associated EL-hyperstructure. Then I is minimal among all left (reps. right) ideals (ordered ideals) of S, which are also an upper end of S if and only if I is minimal among all left (reps. right) hyperideals of $(S, *)$ with the property $(I]_{\leq} = I$.*

Definition 3.26. A non-empty subset I of semihypergroup $(H, \circ)$ is called a *bi-hyperideal* of H if $I \circ H \circ I \subseteq I$. A bi-hyperideal I of H is called a *subhyperidempotent* of H if $I \circ I \subseteq I$.

Theorem 3.27. *Suppose $(S, *)$ is the associated EL-hyperstructure of a quasi-semigroup $(S, \cdot, \leq)$, and I is a (an ordered) bi-ideal of S, which is also an upper end of S. Then,*
1) I is a bi-hyperideal of $(S, *)$.
2) If I is a subidempotent in $(S, \cdot, \leq)$ (i.e. $I \cdot I \subseteq I$), then I is a subhyperidempotent of $(S, *)$ (i.e. $I * I \subseteq I$).

Proof. 1) We have to show that $I * S * I \subseteq I$. Let $x, y \in I$, and $z \in S$. Then $x * z * y = \bigcup_{t \in x * z} t * y = \bigcup_{t \in [x \cdot z)_{\leq}} [t \cdot y)_{\leq}$. Now $x \cdot z \leq t$ implies

$x \cdot z \cdot y \leq t \cdot y$. As I is a bi-ideal of S, there holds $x \cdot z \cdot y \in I$ and since I is an upper end of S, we conclude that $t \cdot y \in I$. This implies that $[t \cdot y)_{\leq} \subseteq I$. Thus $x * z * y \subseteq I$ for all $(x, y) \in I^2$ and $z \in S$.
2)Let x, y be arbitrary elements in I. As I is subidempotent of S, there holds $x \cdot y \in I$ and since I is an upper end of S, we have $x * y = [x \cdot y)_{\leq} \subseteq I$, which implies that $I * I \subseteq I$. □

Example 3.28. Let S = {a,b,c,d,f} be an ordered semigroup with the following multiplication table:

$\cdot$	a	b	c	d	f
a	a	b	c	d	f
b	b	a	c	d	f
c	c	c	c	c	c
d	c	c	c	c	c
f	f	f	c	d	f.

We define the order relation $\leq$ as follows:

$$\leq := \{(d, c), (f, b), (f, c), (f, a), (a, a), (b, b), (c, c), (d, d), (f, f)\}.$$

Consider $I = \{c, d\}$. It is easy to see that I is an ordered bi-ideal of $(S, \cdot, \leq)$, which is also an upper end of S (i.e. $I \cdot S \cdot I \subseteq I$ and $[\{d, c\})_{\leq} = \{d, c\}$). Using the Ends lemma construction, one can achieve the semihypergroup $(S, *)$ using the following table:

$*$	a	b	c	d	f
a	a	b	c	{d,c}	{a,b,c,f}
b	b	a	c	{d,c}	{a,b,c,f}
c	c	c	c	c	c
d	c	c	c	c	c
f	{a,b,c,f}	{a,b,c,f}	c	{d,c}	{a,b,c,f}.

Now, a simple computation shows that I is a bi-hyperideal of $(S, *)$.

Theorem 3.29. *Let $(S, \cdot, \leq)$ be a quasi-ordered monoid, and $(S, *)$ be its associated EL-hyperstructure. Then,*
1) I is a (an ordered) bi-ideal of S, which is also an upper end of S if and only if I is a bi-hyperideal of $(S, *)$ (with the property $(I]_{\leq} = I$).
2) I is a (an ordered) subidempotent of S, which is also an upper end of S if and only if I is a subhyperidempotent of $(S, *)$ (with the property $(I]_{\leq} = I$).

Proof. Regarding Theorem 3.9, Theorem 3.27, and the fact that $a \cdot b \in a * b$ for all $(a, b) \in S^2$, the statements can be easily proved. □

Theorem 3.30. *Suppose $(S,*)$ is the associated EL-hyperstructure of a quasi-semigroup $(S,\cdot,\leq)$, and I is an (ordered) interior ideal of S, which is also an upper end of S. Then, I is an interior hyperideal of $(S,*)$.*

Proof. The proof is similar to the proof of the first part of Theorem 3.27. □

Remark 3.31. Notice that, as mentioned in Remark 3.8, each interior hyperideal of $(S,*)$ is an interior ideal of $(S,\cdot,\leq)$ but it is not necessarily an upper end of S. Also, an arbitrary interior hyperideal of $(S,*)$ is not always an ordered interior ideal of $(S,\cdot,\leq)$. Actually, it is easy to see that:

Theorem 3.32. *Let $(S,\cdot,\leq)$ be a quasi-ordered monoid, and $(S,*)$ be its associated EL-hyperstructure. Then I is an (ordered) interior ideal of S, which is also an upper end of S if and only if I is an interior hyperideal of $(S,*)$ (with the property $(I]_{\leq}=I$).*

Definition 3.33. Let $(H,\circ)$ be a semihypergroup. A non-empty subset I of H is called a (m,n)*-hyperideal* of H if $I^m\circ S\circ I^n\subseteq I$.

Theorem 3.34. *Suppose $(S,*)$ is the associated EL-hyperstructure of a quasi semigroup $(S,\cdot,\leq)$, and I is an (m,n)-ideal (ordered (m,n)-ideal) of S, which is also an upper end of S. Then, I is an (m,n)-hyperideal of $(S,*)$.*

Proof. Suppose $x\in I_*^m * S * I_*^n$, in which I_*^m is equal to $I*I*\cdots*I$ (m times). There exit $a\in I_*^m$, $s\in S$, and $b\in I_*^n$ such that $x\in a*s*b=\bigcup\limits_{t\in s*b} a*t=\bigcup\limits_{t\in[s\cdot b)_{\leq}}[a\cdot t)_{\leq}$. Thus there exists $t_1\in[s\cdot b)_{\leq}$ (i.e. $s\cdot b\leq t_1$) such that $x\in[a\cdot t_1)_{\leq}$ (i.e. $a\cdot t_1\leq x$). Now, the relation $s\cdot b\leq t_1$ implies that $a\cdot s\cdot b\leq a\cdot t_1\leq x$. On the other hand, since $a\in I_*^m$ and $b\in I_*^n$, there exist $(x_1,x_2,\cdots,x_m)\in I^m$ and $(y_1,y_2,\cdots,y_n)\in I^n$ such that $x_1\cdot x_2\cdots x_m\leq a$ and $y_1\cdot y_2\cdots y_n\leq b$. This implies that $x_1\cdot x_2\cdots x_m\cdot s\cdot y_1\cdot y_2\cdots y_n\leq a\cdot s\cdot b\leq x$, which implies that $x\in[x_1\cdot x_2\cdots x_m\cdot s\cdot y_1\cdot y_2\cdots y_n)_{\leq}$ where $x_1\cdot x_2\cdots x_m\cdot s\cdot y_1\cdot y_2\cdots y_n$ is in $I^m\cdot S\cdot I^n\subseteq I$. Finally, since I is an upper end of S, there holds $[x_1\cdot x_2\cdots x_m\cdot s\cdot y_1\cdot y_2\cdots y_n)_{\leq}\subseteq I$, and consequently, $x\in I$. □

In the next two examples, we show that quasi-ideals of an ordered semigroup $(S,\cdot,\leq)$ would not be quasi-hyperideal of the associated EL-hyperstructure $(S,*)$.

Example 3.35. Consider the ordered semigroup $(S,\cdot,\leq)$ mentioned in Example 3.14 and its associated EL-hyperstructure $(S,*)$. Quasi-ideals of S are: {a}, {a,b}, {a,c}, {a,d}, {a,f}, {a,b,d}, {a,c,d}, {a,b,f}, {a,c,f}, and S (see [19]). Now, consider the quasi-ideal $I = \{a\}$. There holds $a * S = S * a = S$, which implies that $I * S \bigcap S * I \nsubseteq I$ i.e. $I = \{a\}$ is not a quasi-hyperideal of S. A similar proof holds for other quasi-ideals of S.

Example 3.36. Let $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, and $S = M_2(\mathbb{N}_0)$. Then S with a usual matrix multiplication is a semigroup. Define the relation $\leq$ on S as follows: for all $A, B \in S$ and $i, j \in \{1, 2\}$, $A \leq B \iff a_{ij} \leq b_{ij}$. It is easy to check that $(S, \cdot, \leq)$ is a partially-ordered semigroup. In addition, let $I = \{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} |a \in \mathbb{N}_0\} \subseteq S$, which is an ordered quasi-ideal of $(S, \cdot, \leq)$ (i.e. $IS \bigcap SI \subseteq I$). We claim that I can not be a quasi-hyperideal in $(S, *)$. Indeed, there holds $I * S = \bigcup_{X \in I, Y \in S} X * Y = \bigcup_{x,a \in \mathbb{N}_0} [\begin{bmatrix} xa & 0 \\ 0 & 0 \end{bmatrix})_{\leq} = S$, and similarly, $S * I = S$. So, $I * S \bigcap S * I = S \nsubseteq I$.

Definition 3.37. Let $(H, \circ)$ be a (semi)hypergroup, and P be a hyperideal of $(H, \circ)$. The hyperideal P is called *semiprime* if $I \circ I \subseteq P$ implies that $I \subseteq P$ for all hyperideal I of $(H, \circ)$.

Example 3.38. Let S = {a,b,c} be the semihypergroup with the following table:

∘	a	b	c
a	a	a	a
b	a	c	a,b
c	a	a,b	a,c

One can easily check that $P = \{a\}$ is a semiprime hyperideal of $(S, \circ)$.

Theorem 3.39. *Suppose $(S, *)$ is the associated EL-hyperstructure of a quasi-semigroup $(S, \cdot, \leq)$, and P is a (an ordered) semiprime ideal of S, which is also an upper end of S. Then P is a semiprime hyperideal of $(S, *)$ (with the property $(P]_{\leq} = P$).*

Proof. Suppose I is an arbitrary hyperideal of $(S, *)$ with the property $I * I \subseteq P$. We have to show that $I \subseteq P$. Let $x \in I$. Then, $x * x \subseteq I * I \subseteq P$. On the other hand, by the reflexive property of order $\leq$, $x^2 = x \cdot x \in [x \cdot x)_{\leq} = x * x \subseteq P$, which implies that $x \in P$. So, $I \subseteq P$, and finally, P is a semiprime hyperideal of $(S, *)$. A similar proof holds for ordered semiprime ideals of semigroup $(S, \cdot, \leq)$ and semiprime hyperideal of semihypergroup $(S, *)$ with the property $(P]_{\leq} = P$. $\square$

Definition 3.40. Let $(H, \circ)$ be a (semi)hypergroup, and P be a hyperideal of $(H, \circ)$. The hyperideal P is called *prime* if $I \circ J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$ for all hyperideal I and J of $(H, \circ)$.

Example 3.41. Let $S = (\mathbb{N}, \otimes)$, in which $m \otimes n = \{kmn | k \in \mathbb{N}\}$ for all $(m, n) \in \mathbb{N}^2$. Then, an easy verification shows that $I = (2)$ is a prime hyperideal.

Theorem 3.42. *Suppose $(S, *)$ is the associated EL-hyperstructure of a quasi-semigroup $(S, \cdot, \leq)$, and P is a (an ordered) prime ideal of S, which is also an upper end of S. Then, P is a prime hyperideal of $(S, *)$ (with the property $(P]_{\leq} = P$).*

Proof. At first, let P be an ordinary prime ideal of S. Suppose I_1 and I_2 are two hyperideals of $(S, *)$ such that $I_1 * I_2 \subseteq P$. Now, by Remark 3.8, I_1 and I_2 are two ideals of semigroup S. On the other hand, there holds $I_1 \cdot I_2 \subseteq I_1 * I_2 \subseteq P$, which implies that $I_1 \subseteq P$ or $I_2 \subseteq P$. A similar proof holds for ordered prime ideals of $(S, \cdot, \leq)$ and prime hyperideals of $(S, *)$ with the property $(P]_{\leq} = P$. □

Definition 3.43. Let $(H, \circ)$ be a semihypergroup. An element $x \in H$ is called an *idempotent* if $x \in x \circ x$. We denote the set of all idempotent of H by $E(H)$. In addition, we define the semihypergroup $(H, \circ)$ as a *idempotent semihypergroup* or a *hyperband* if $E(H) = H$.

Example 3.44. Let S = {a,b,c} be the semigroup with the following multiplication table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c.

We define the order relation $\leq$ as follows:

$$\leq := \{(a, c), (a, b), (a, a), (b, b), (c, c)\}.$$

It is not difficult to see that $(S, \cdot, \leq)$ is an ordered semigroup, in which $E(S) = S$ (i.e. each element is an idempotent). Using the Ends lemma construction, one can achieve EL-hyperstructure $(S, *)$ as follows:

$*$	a	b	c
a	S	S	S
b	S	{b}	S
c	S	S	{c}.

As it can be seen, for all $x \in S$, there holds $x \in x * x$. Thus $E(S) = S$ (i.e.$(S, *)$ is a hyperband).

In the next Theorem, we show that the associated EL-hyperstructure of an ordered band (an ordered semigroup in which each element is an idempotent) is always a hyperband.

Theorem 3.45. *Let $(S, \cdot, \leq)$ be a quasi-semigroup, and also a band and $(S, *)$ be its associated EL-hyperstructure. Then $(S, *)$ is a hyperband.*

Proof. Due to reflexivity of the relation $\leq$, for all $x \in S$, there holds $x \in [x)_{\leq} = [x \cdot x)_{\leq} = x * x$, i.e. x is an idempotnet. □

Remark 3.46. The converse of Theorem 3.45 is not true. The following example shows that the associated EL-semihypergroup of an ordered semigroup $(S, \cdot, \leq)$ with the property $E(S) \neq S$ can be a hyperband.

Example 3.47. Let $(S, \cdot, \leq)$ be the ordered semigroup defined in Example 5. It is clear that $E(S) \neq S$, i.e. $(S, \cdot)$ is not a band. However, its associated EL-semihypergroup is a hyperband.

4. Ends lemma based Γ-semihypergroup

Now, we construct a Γ-semihypergroup from a partially-ordered Γ-semigroup. At first, we recall some definitions and examples that can be found in [8] and [22].

Definition 4.1. [1] Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exists a map $S \times \Gamma \times S \longrightarrow S$, written (x, γ, x) by $x\gamma y$, satisfying the identities $(a\gamma b)\delta c = a\gamma(b\delta c)$ for all $(a, b, c) \in S^3$ and $(\gamma, \delta) \in \Gamma^2$.

Example 4.2. For $a, b \in [0, 1]$, let $M = [0, a]$, and $\Gamma = [0, b]$. Then M is a Γ-semigroup under usual multiplication.

Example 4.3. Let S be an arbitrary semigroup, and Γ be any non-empty set. For all $(x, y) \in S^2$ and $\gamma \in \Gamma$, define $a\gamma b = ab$. It can easily be checked that S is a Γ-semigroup.Thus a semigroup can be regarded as a Γ-semigroup.

Definition 4.4. A partially-ordered Γ-semigroup is a partially-ordered set $(S, \leq)$, in which S is a Γ-semigroup, and the relation $x \leq y$ implies that $x\gamma z \leq y\gamma z$ and $z\gamma x \leq z\gamma y$ for all $(x, y, z) \in S^3$ and $\gamma \in \Gamma$.

Definition 4.5. Let S and Γ be two non-empty sets. Then, S is called a Γ-semihypergroup if each $\gamma \in \Gamma$ is a hyperoperation on S, i.e., $x\gamma y \subseteq S$ for all $(x, y) \in S^2$, and for every $(\alpha, \beta) \in \Gamma^2$ and $(x, y, z) \in S^3$ there holds $x\alpha(y\beta z) = (x\alpha y)\beta z$ (associative property).

Example 4.6. Let S be a non-empty, set and Γ be a non-empty subset of S. Define $x\gamma y = \{x, \gamma, y\}$ for all $x, y \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ-semihypergroup.

Theorem 4.7. *Suppose* $(S, \cdot, \leq)$ *is a partially-ordered* Γ*-semigroup. Then, its EL-hyperstructure is a* Γ*-semihypergroup.*

Proof. We must show that (1) every $\gamma \in \Gamma$ is hyperoperation on S, and (2) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for each $(x, y, z) \in S^3$ and $(\alpha, \beta) \in \Gamma^2$.
(1) Let γ be an arbitrary element in Γ. For each $(x, y) \in S^2$, define

$$x\gamma y = [x \cdot \gamma \cdot y)_{\leq} = \{s \in S | x \cdot \gamma \cdot y \leq s\} \subseteq S.$$

Clearly, γ is a hyperoperation on S.
(2) To prove the associativity, we first show that $\bigcup\limits_{s \in [x \cdot \alpha \cdot y)_{\leq}} [s \cdot \beta \cdot z)_{\leq} = \bigcup\limits_{t \in [y \cdot \beta \cdot z)_{\leq}} [x \cdot \alpha \cdot t)_{\leq}$ for each $(x, y, z) \in S^3$ and $(\alpha, \beta) \in \Gamma^2$. Let $m \in \bigcup\limits_{t \in [y \cdot \beta \cdot z)_{\leq}} [x \cdot \alpha \cdot t)_{\leq}$. There exists $t_0 \in [y \cdot \beta \cdot z)_{\leq}$ such that $m \in [x \cdot \alpha \cdot t_0)_{\leq}$ (i.e. $x \cdot \alpha \cdot t_0 \leq m$). However, $y \cdot \beta \cdot z \leq t_0$ implies that $x \cdot \alpha \cdot y \cdot \beta \cdot z \leq x \cdot \alpha \cdot t_0 \leq m$. Set $x \cdot \alpha \cdot y = s_0 \in S$. Then, due to reflexivity of $\leq$, there holds $x \cdot \alpha \cdot y = s_0 \in [s_0)_{\leq}$ and $s_0 \cdot \beta \cdot z \leq m$. This means that $m \in [s_0 \cdot \beta \cdot z)_{\leq} \subseteq \bigcup\limits_{s \in [x \cdot \alpha \cdot y)_{\leq}} [s \cdot \beta \cdot z)_{\leq}$. Consequently, we have

$$\bigcup_{s \in [x \cdot \alpha \cdot y)_{\leq}} [s \cdot \beta \cdot z)_{\leq} \subseteq \bigcup_{t \in [y \cdot \beta \cdot z)_{\leq}} [x \cdot \alpha \cdot t)_{\leq}.$$

With a similar proof, we can see $\supseteq$. Finally,

$$(x\alpha y)\beta z = \bigcup_{s \in [x \cdot \alpha \cdot y)_{\leq}} [s \cdot \beta \cdot z)_{\leq} = \bigcup_{t \in [y \cdot \beta \cdot z)_{\leq}} [x \cdot \alpha \cdot t)_{\leq} = x\alpha(y\beta z).$$

□

5. Conclusions and future work

For almost all algebraic (hyper)structures, the set of sub(hyper) structures (such as subspaces of a vector space, submodules of a module, and ideals of semigroup or a ring and so on) plays a vital and important role in studying and classifying their based (hyper) structures. In this work, we studied different hyperideals of an EL-semihypergroup associated to a quasi-ordered semigroup. Also in the last section, we extended the concept of EL- hyperstructures to EL-Γ-hyperstructures.

The generalization of End lemma and EL-hyperstructures can be carried out in different viewpoints. For example, Ghazavi et al, in [7], started from a partially/quasi-ordered semihypergroup (instead of a partially/quasi-ordered semigroup), and introduced EL^2- hyperstructures, and investigated the similarities and differences between EL and EL^2-hyperstructures. Another idea to do this (extension of End lemma) is to use ternary and n-ary relations instead of binary relations in the primitive concept of Ends lemma, on which we are now working on.

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IDEALS IN EL-SEMIHYPERGROUPS ASSOCIATED TO ORDERED SEMIGROUPS

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ایدآل‌ها در EL- ابرنیم‌گروه‌های وابسته به نیم‌گروه‌های مرتب

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دراین مقاله رابطه بین انواع مختلف ایدآل‌های یک نیم‌گروه مرتب مثل $(S, \cdot, \leq)$ همانند ایدآل‌های اول، مینیمال، ماکسیمال، (m, n)-ایدآل‌ها، شبه‌ایدآل‌ها و ... را با ابر ایدآل‌های متناظر در El-ابرنیم‌گروه وابسته یعنی $(S, *)$ (در صورت وجود) بررسی می‌کنیم. در ادامه با استفاده از ساختار لم پایانی کلاس Γ-EL-ابرنیم‌گروه‌ها را از روی یک Γ-نیم‌گروه مرتب می‌سازیم.

کلمات کلیدی: El- ابرساختارها، لم پایانی، شبه ایدآل، ایدآل داخلی، (m, n)-ایدآل، نیم‌گروه مرتب