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ON ABSOLUTE CENTRAL AUTOMORPHISMS FIXING THE CENTER ELEMENTWISE

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Abstract. Let G be a finite p-group. In this work we give the necessary and sufficient conditions on G such that each absolute central automorphism of G fixes the center element-wise. Also we classify all groups of the orders p^3 and p^4, whose absolute central automorphisms fix the center element-wise.

1. Introduction

Let G be a group. Our notations are standard. For example, G', $L(G)$, and $\exp(G)$ denote the commutator subgroup, absolute center, and exponent of G, respectively. Let $\mathrm{cl}(G)$ denote the nilpotency class of G. A non-abelian group G of order p^n is of maximal class if $\mathrm{cl}(G) = n-1$. Also we use the notation $G^{p^n} = \langle g^{p^n} | g \in G\rangle$.

An automorphism α of G is called a central automorphism if $x^{-1}\alpha(x) \in Z(G)$ for each $x \in G$. The set of all central automorphisms of group G, denoted by $\mathrm{Aut}_c(G)$, fix G' element-wise. Hegarty, in [1], generalized the concept of center into absolute center. Also he introduced the absolute central automorphisms. An automorphism γ of G is called an absolute central automorphism if it induces the identity on the factor group $G/L(G)$, or equivalently, $x^{-1}\gamma(x) \in L(G)$ for each $x \in G$. Let us denote the set of all absolute central automorphisms of G by $\mathrm{Aut}_l(G)$. $\mathrm{Aut}_l(G)$ is a normal subgroup of the full automorphism group of G, contained in $\mathrm{Aut}_c(G)$.

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Attar, in [5], and Jafari, in [2], gave the necessary and sufficient conditions on a finite p-group G such that $\mathrm{Aut}_c(G) = C_{Aut_c(G)}(Z(G))$. In this paper, we intend to give the necessary and sufficient conditions on p-group G, in which $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$, where $C_{Aut_l(G)}(Z(G))$ is the group of all absolute central automorphisms of G fixing $Z(G)$ element-wise.

2. Preliminary results

We first state some results that will be used in the proof of the main theorem.

Let G be a group. For each element, $g \in G$, and $\alpha \in \mathrm{Aut}(G)$, $[g, \alpha] = g^{-1}\alpha(g)$ is the *autocommutator* of g and α.

Definition 2.1. Let G be a group. The *absolute center* $L(G)$ of G is defined by:

$$L(G) = \{g \in G \mid [g, \alpha] = 1,\ \forall \alpha \in \mathrm{Aut}(G)\}.$$

Clearly, it is a characteristic subgroup of G and $L(G) \leq Z(G)$.
Likewise,

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1,\ \forall \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)\},$$

stands for the n^{th}-*absolute center* of G.

Definition 2.2. A group G is called the *autonilpotent* of class n if n is the smallest natural number such that $L_n(G) = G$.

Lemma 2.3. [4, Lemma 2.11] *If G is a finite autonilpotent group of class 2, then $\mathrm{Aut}_l(G) = \mathrm{Aut}(G)$.*

Proposition 2.4. [4, Proposition 2.12] *If G is a finite autonilpotent group of class 2, then $G/L(G)$ is abelian.*

Lemma 2.5. [3, Corollary 3.7] *Let G be a non-abelian finite p-group. Then $L(G) \leq \Phi(G)$.*

3. Main results

Let G be a finite p-group, and let $\alpha \in Aut_l(G)$ and $p^n = \exp(L(G))$. Since $g^{-1}\alpha(g) \in L(G)$, $\alpha(g) = gl$ for some $l \in L(G)$. Thus $\alpha(g^{p^n}) = g^{p^n} l^{p^n} [l, g]^{\binom{p^n}{2}}$. Now since $L(G) \subseteq Z(G)$, $[l, g] = 1$. Also $l^{p^n} = 1$. Therefore, $\alpha(g^{p^n}) = g^{p^n}$, for every $g \in G$.

Theorem 3.1. *Let G be a non-abelian finite p-group. Then $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$ if and only if $Z(G)G' \subseteq G'L(G)G^{p^n}$, where $p^n = \exp(L(G))$.*

Proof. Suppose $Z(G)G' \subseteq G'L(G)G^{p^n}$, where $p^n = \exp(L(G))$. We know $C_{Aut_l(G)}(Z(G)) \leq \mathrm{Aut}_l(G)$. Now assume that $\alpha \in \mathrm{Aut}_l(G)$, and $x \in Z(G)$. We can write $x = abg^{p^n}$ for some $a \in G'$, $b \in L(G)$, and $g \in G$. According to the previously-mentioned points, $\alpha(g^{p^n}) = g^{p^n}$ and $\alpha(b) = b$. Also $\mathrm{Aut}_l(G)$ acts trivially on G'. Hence, $\alpha(x) = x$ and so $\alpha \in C_{Aut_l(G)}(Z(G))$. This shows that $\mathrm{Aut}_l(G) \subseteq C_{Aut_l(G)}(Z(G))$, and whence $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$.

To prove the converse, assume that $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$, and $Z(G)G' \nsubseteq G'L(G)G^{p^n}$. Thus exists $x \in Z(G)$, which is not in $G'L(G)G^{p^n}$. Let $G/G'L(G) = \langle x_1G'L(G)\rangle \times \cdots \times \langle x_kG'L(G)\rangle$, where $x_1, x_2, \ldots, x_k \in G$. Therefore, $xG'L(G) = x_1^{p^{t_1}}G'L(G)\ldots x_k^{p^{t_k}}G'L(G)$ for some $t_1, ..., t_k$. Since $x \notin G'L(G)G^{p^n}$, then $x_i^{p^{t_i}} \notin G^{p^n}$, and so $p^{t_i} < p^n$ for some i. Now select $l \in L(G)$, where $O(l) = min(p^n, O(x_iG'L(G)))$, and define $f : G/G'L(G) \longrightarrow L(G)$ by $x_iG'L(G) \mapsto l$ and $x_jG'L(G) \mapsto 1$, for $j \neq i$. Then f can be considered as a homomorphism. Now, consider the map $\sigma_f : G \longrightarrow G$ defined by $\sigma_f(a) = af(aG'L(G))$. Clearly, σ_f is an endomorphism of G. Now suppose that $x \in Ker(\sigma_f)$. Then $f(xG'L(G)) = x^{-1}$. Also σ_f acts trivially on elements of $L(G)$, so we can write $x^{-1} = \sigma_f(x^{-1}) = x^{-1}f(x^{-1}G'L(G)) = x^{-1}x = 1$. Therefore, $x = 1$. This shows that σ_f is one-to-one, and since G is finite, one can see that the homomorphism σ_f is a bijection. Hence, σ_f is an absolute central automorphism of G. Moreover, $f(xG'L(G)) = f(x_1^{p^{t_1}}G'L(G)\ldots x_k^{p^{t_k}}G'L(G))$, and so $f(xG'L(G)) = f(x_i^{p^{t_i}}G'L(G)) = l^{p^{t_i}}$. Since, $p^{t_i} < p^n$, therefore, $l^{p^{t_i}}$ is a non-trivial element of $L(G)$. Hence, $\sigma_f \notin C_{Aut_l(G)}(Z(G))$, which is a contradiction. □

Corollary 3.2. *Let G be a non-abelian finite p-group, and $\exp(L(G)) = p$. Then $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$ if and only if $Z(G) \subseteq \Phi(G)$.*

Proof. By using Theorem 3.1 and Lemma 2.5, it is clear. □

Corollary 3.3. *Let G be a finite autonilpotent group of class 2. Then $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$ if and only if $Z(G) = L(G)G^{p^n}$, where $p^n = \exp(L(G))$.*

Proof. Suppose $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$. By the Theorem 3.1 and Proposition 2.4, $Z(G) \subseteq L(G)G^{p^n}$. Also since $G' \subseteq L(G)$, for every $a, b \in G$, we have $[a, b]^{p^n} = 1$, and whence $[a^{p^n}, b] = 1$. This means that for every $a \in G$, $a^{p^n} \in Z(G)$ and $G^{p^n} \leq Z(G)$. Therefore, $L(G)G^{p^n} \subseteq Z(G)$, and so $Z(G) = L(G)G^{p^n}$. The converse holds by Theorem 3.1. □

Corollary 3.4. *Let G be a finite autonilpotent group of class* 2 *and* $\exp(L(G)) = p^n$. *If $Z(G) = L(G)G^{p^n}$, then each automorphism of G fixes the center element-wise.*

Proof. It follows from Lemma 2.3 and Corollary 3.3. □

4. Absolute central automorphism of groups of orders p^3 and p^4

Now we classify all groups G of the orders p^3 and p^4, whose absolute central automorphism of G fix the center element-wise.

Lemma 4.1. *Let G be a group of order p^n of maximal class. Then* $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$.

Proof. For each p-group of maximal class, we have $Z(G) \leq G'$. Hence, these groups satisfy $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$. □

Corollary 4.2. *For each non-abelian group G of order p^3,* $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$.

Proof. Let G be a non-abelian group of order p^3. Then $\mathrm{cl}(G) = 2$, and by Lemma 4.1, $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$. □

Proposition 4.3. *Let G be a non-abelian group of order p^4. Then* $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$, *except when both $L(G) \cong C_p$ and $Z(G) \nsubseteq \Phi(G)$ do occur.*

Proof. Suppose $|G| = p^4$. Then G is nilpotent of class at most 3. Since G is non-abelian, so $\mathrm{cl}(G) = 3$ or 2. If $\mathrm{cl}(G) = 3$, by Lemma 4.1, G satisfies $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$. Now suppose that $\mathrm{cl}(G) = 2$. Since G is not an extra-special p-group, we have $|Z(G)| \neq p$. Hence, $|Z(G)| = p^2$. Thus $|L(G)| = 1, p$ or p^2. If $|L(G)| = 1$, then $\mathrm{Aut}_l(G) = \langle 1 \rangle$, and so G satisfies $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$. If $|L(G)| = p$, then $\exp(L(G)) = p$, and by Corollary 3.2, $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$ if and only if $Z(G) \subseteq \Phi(G)$. Thus in this state, when $Z(G) \nsubseteq \Phi(G)$, then $\mathrm{Aut}_l(G) \neq C_{Aut_l(G)}(Z(G))$. Finally, let $|L(G)| = p^2$. Then $L(G) = Z(G)$, and hence, $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$. Therefore, in all states, $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$, except when both $L(G) \cong C_p$ and $Z(G) \nsubseteq \Phi(G)$ do occur. □

Proposition 4.4. *Using* GAP [6] *and the previous results, the only non-abelian groups G of order* 16 *such that* $\mathrm{Aut}_l(G) = C_{Aut_l(G)}(Z(G))$, *are* $D_{16} = \langle x, y \mid x^8 = y^2 = 1, y^{-1}xy = x^{-1} \rangle$, $Q_{16} = \langle x, y \mid x^8 = 1, x^4 = y^2, y^{-1}xy = x^{-1} \rangle$, $S_{16} = \langle x, y \mid x^8 = y^2 = 1, y^{-1}xy = x^3 \rangle$, $\langle x, y \mid x^4 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$, $\langle x, y \mid x^4 = y^4 = (xy)^2 = (xy^{-1})^2 = 1 \rangle$, $\langle x, y \mid x^2 = y^8 = 1, x^{-1}yx = y^{-3} \rangle$.

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بررسی خودریختی‌های مرکزی مطلق که مرکز گروه را نقطه‌وار ثابت نگه می‌دارند

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فرض کنیم G یک p- گروه متناهی باشد. ما در این مقاله شرط لازم و کافی برای گروه G فراهم می‌کنیم به‌طوری‌که هر خودریختی مرکزی مطلق از G مرکز را نقطه‌وار ثابت نگه دارد. هم‌چنین ما تمام گروه‌ها از مرتبه‌ی p^3 و p^4 را که خودریختی‌های مرکزی مطلق آن‌ها مرکز را ثابت نگه می‌دارند، دسته‌بندی می‌کنیم.

کلمات کلیدی: مرکز مطلق، خودریختی‌های مرکزی مطلق، p-گروه‌های متناهی .