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RESULTS ON ALMOST COHEN-MACAULAY MODULES

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Abstract. Let $(R, \underline{m})$ be a commutative Noetherian local ring, and M be a non-zero finitely-generated R-module. We show that if R is almost Cohen-Macaulay and M is perfect with finite projective dimension, then M is an almost Cohen-Macaulay module. Also, we give some necessary and sufficient conditions on M to be an almost Cohen-Macaulay module, by using Ext functors.

1. Introduction

We shall assume throughout this note that R is a commutative Noetherian ring with non-zero identity, and M is a non-zero finitely-generated R-module. The projective dimension of an R-module M is denoted by $\operatorname{pd} M$. The well-known notion, $\operatorname{grade} M$, has been introduced by Rees, in [9], as the least integer $n \geq 0$ such that $\operatorname{Ext}^n_R(M, R) \neq 0$. Foxby, in [3, Proposition 2.1(h)], defined the $\operatorname{grade}_N M$ as the least integer $n \geq 0$ such that $\operatorname{Ext}^n_R(M, N) \neq 0$, where N is a non-zero finitely-generated R-module. An R-module M is perfect if $\operatorname{pd} M = \operatorname{grade} M$, see [1, Definition 1.4.15].

Han, in [4], and later Kang, in [6] and [7], defined that an R-module M is almost Cohen-Macaulay (i.e. aCM) if $\operatorname{grade}(P, M) = \operatorname{depth} M_P$ for every $P \in \operatorname{Supp}(M)$. Also, R is called an aCM ring if it is an aCM module when it is regarded as a module over itself. In [6], Kang gave some fundamental properties and some characterizations of aCM modules and in [7]. He gave some interesting examples of aCM modules.

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In [2], Chu et al., by using the first non-vanishing local cohomology module, gave a necessary and sufficient condition for an R-module M to be an aCM module. Also Ionescu, in [5], studied the behavior of aCM rings with respect to flat morphisms.

In this work, we prove that if R is an aCM ring, and M is a perfect module with finite projective dimension. Then M is an aCM module. Also we examine the behavior of aCM moules with respect to flat morphisms. Moreover, by using Ext functors, we give a necessary and sufficient condition for an R module M to be an aCM module. For basic definitions, we refer the reader to [1] or [8].

2. Some basic results on aCM modules

Lemma 2.1. *Let $(R, \underline{m})$ be a local ring, and M be an aCM R-module. Then for any $P \in \operatorname{Ass}(M)$, we have $\dim M - \dim R/P \leq 1$.*

Proof. From [1, Proposition 1.2.13], we have $\operatorname{depth} M \leq \dim R/P$ for all $P \in \operatorname{Ass}(M)$, and on the other hand, have $\dim R/P \leq \dim M$. Since M is an aCM module, it, therefore, follows that $\dim M \leq \operatorname{depth} M + 1$, and this completes the proof. □

Lemma 2.2. *Let $(R, \underline{m})$ be a local ring. If for any proper ideal $\underline{a}$ of R and any $P \in \operatorname{Supp}(M/\underline{a}M)$ we have $\operatorname{grade}(\underline{a}, M) = \operatorname{grade}(\underline{a}R_P, M_P)$, then M is aCM.*

Proof. The proof is clear. □

Theorem 2.3. *Let $(R, \underline{m})$ be a local ring, and M be an aCM R-module. Then $\dim M - \dim M/\underline{a}M \leq \operatorname{grade}(\underline{a}, M) + 1$ for all ideal $\underline{a} \subseteq \underline{m}$.*

Proof. If $\operatorname{grade}(\underline{a}, M) = 0$, then there exists $P \in Ass(M)$ with $\underline{a} \subseteq P$, and, therefore, $\dim R/P \leq \dim M/\underline{a}M$. Thus by using Lemma 2.1, $\dim M - \dim M/\underline{a}M \leq 1$. If $\operatorname{grade}(\underline{a}, M) > 0$, then there exists $x \in \underline{a}$, which is regular on M. One has $\operatorname{grade}(\underline{a}, M/xM) = \operatorname{grade}(\underline{a}, M) - 1$ and $\dim M/xM = \dim M - 1$, so that induction completes the argument. □

Corollary 2.4. *Let $(R, \underline{m})$ be a local ring, and M be an aCM R-module. Then $\dim M - \dim M/PM \leq \dim M_P + 1$ for all $P \in \operatorname{Supp}(M)$.*

Proof. By Theorem 2.3, $\dim M - \dim M/PM \leq \operatorname{grade}(P, M) + 1$. Since $\operatorname{grade}(P, M) \leq \operatorname{ht} P = \dim M_P$, the inequality follows. □

The following result easily follows by Theorem 2.3.

Corollary 2.5. *Let $(R, \underline{m})$ be a local ring, and I be a proper ideal of R. If R is aCM, then $\operatorname{ht} I \leq \dim R - \dim R/I \leq \operatorname{ht} I + 1$.*

The following theorem extends [10, 1.9].

Theorem 2.6. *Let R be an aCM ring, and M be a perfect module with finite projective dimension. Then M is an aCM module.*

Proof. For any $P \in \operatorname{Supp}(M)$, one has the inequalities $\operatorname{grade} M \leq \operatorname{grade} M_P \leq \operatorname{pd} M_P \leq \operatorname{pd} M$. Therefore, if M is a perfect module, then M_P is a perfect R_P-module. Thus, we may assume that R is a local ring. The Auslander-Buchsbaum formula gives $\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R$, and Corollary 2.5 yields $\operatorname{grade} M + \dim M \leq \dim R$. Thus $\dim M - \operatorname{depth} M \leq \dim R - \operatorname{depth} R \leq 1$, and so M is an aCM module. □

The following result extends [5, Proposition 2.2].

Proposition 2.7. *Let $\varphi : (R, \underline{m}) \longrightarrow (S, \underline{n})$ be a local homomorphism of Noetherian local rings. Suppose M is a finitely-generated R-module, and N is an R-flat finitely-generated S-module. If $M \otimes_R N$ is an aCM S-module, then M is an aCM R-module and $N/\underline{m}N$ is an aCM S-module. Moreover, if the R-module M and the S-module $N/\underline{m}N$ are aCM and one of them is CM, then the S-module $M \otimes_R N$ is aCM.*

Proof. By [1, Proposition 1.2.16(a) and Theorem A.11(b)] we have $\dim_S(M \otimes_R N) = \dim_R M + \dim_S(N/\underline{m}N)$ and $\operatorname{depth}_S(M \otimes_R N) = \operatorname{depth}_R M + \operatorname{depth}_S(N/\underline{m}N)$. Therefore, $0 \leq \dim_S(M \otimes_R N) - \operatorname{depth}_S (M \otimes_R N) = \dim_R M - \operatorname{depth}_R M + \dim_S(N/\underline{m}N) - \operatorname{depth}_S(N/\underline{m}N) \leq 1$, and so $\dim_R M - \operatorname{depth}_R M \leq 1$ and $\dim_S(N/\underline{m}N) - \operatorname{depth}_S(N/\underline{m}N) \leq 1$. Thus M over R and $N/\underline{m}N$ over S are aCM. Moreover, the above inequality yields the remainder. □

3. A aCM modules and Ext functors

Throughout this section, $(R, \underline{m})$ is a CM local ring of dimension d, and C is a canonical module of R. Recall that a maximal Cohen-Macaulay module C of type 1 and of finite injective dimension is called a canonical module of R.

Proposition 3.1. *Let M be an aCM R-module with* $\operatorname{depth} M = t$. *If* $\dim M - \operatorname{depth} M = 1$, *then* $\operatorname{Ext}^i_R(M, C) \neq 0$ *only if* $i = d - t, d - t - 1$.

Proof. [3, Propositions 3.1(b)] yields that $\operatorname{grade}_C M = \operatorname{grade} M$. Thus [3, Proposition 1.2(g), (i)] implies that $\sup\{i : \operatorname{Ext}^i_R(M, C) \neq 0\} = d - t$ and $\inf\{i : \operatorname{Ext}^i_R(M, C) \neq 0\} = d - t - 1$. This completes the proof. □

Theorem 3.2. *Suppose that M is not a CM R-module. Then M is an aCM module with* $\operatorname{depth} M = t$ *if and only if* $\operatorname{Ext}^i_R(M, C) \neq 0$ *exactly when* $i = d - t, d - t - 1$.

Proof. ($\Longrightarrow$). This is obvious by Proposition 3.1.
($\Longleftarrow$). Since $\mathrm{Ext}_R^{d-t}(M,C) \neq 0$ and $\mathrm{Ext}_R^i(M,C) = 0$ for all $i > d-t$, by [3, Proposition 1.2(g)] we have depth $M = t$. Again, by [3, Propositions 1.2(h) and 3.1(b)] grade $M = d-t-1$, and so by [3, Proposition 1.2(i)] $\dim M = t+1$. Therefore, M is an aC.M module. □

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RESULTS ON ALMOST COHEN-MACAULAY MODULES

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نتایجی در مورد مدول‌های تقریبا کوهن-مکولی

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فرض کنید $(R, \underline{m})$ یک حلقه جابه‌جایی، نوتری و موضعی و M یک R-مدول با تولید متناهی و ناصفر باشد. در این مقاله نشان می‌دهیم اگر R حلقه تقریباً کوهن مکولی و M مدول کامل با بعد پروژکتیوی متناهی باشد آن گاه M تقریباً کوهن مکولی است. ما همچنین با استفاده از فانکتور Ext، یک شرط لازم و کافی برای آن که M تقریباً کوهن-مکولی باشد را ارائه می‌دهیم.

کلمات کلیدی: مدول تقریباً کوهن-مکولی، مدول کامل، فانکتور Ext.