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RATIONAL CHARACTER TABLE OF SOME FINITE GROUPS

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Abstract. The aim of this work is to compute the rational character tables of the dicyclic group T_{4n}, the groups of the orders pq and pqr. Some general properties of rational character tables are also considered.

1. Introduction

In this section, we establish some basic notation, and terminologies that are used throughout this article. Let p be a prime number, and q be a positive integer such that $q|p-1$. Define the group $F_{p,q}$ to be presented by

$$F_{p,q} = \langle a, b : \; a^p = b^q = 1, b^{-1}ab = a^u \rangle,$$

where u is an element of order q in the multiplicative group $\mathbb{Z}_p^*$ [5, Page 290]. In the case that q is also a prime, we use the notation $T_{p,q}$ as $F_{p,q}$. It is easy to see that $F_{p,q}$ is a Frobenius group of the order pq.

Hölder [3] classified groups of order pqr; p, q, and r are the primes. Using his results, we can prove that all groups of the order pqr $(p > q > r)$ are isomorphic to one of the following groups:

- $G_1 = \mathbb{Z}_{pqr}$,
- $G_2 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$,
- $G_3 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$,
- $G_4 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$,
- $G_5 = F_{p,qr}(qr|p-1)$,

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- $G_{i+5} = \langle\ a, b, c\ :\ a^p = b^q = c^r = 1,\ ab = ba,\ c^{-1}bc = b^u,\ c^{-1}ac = a^{v^i}\rangle$, where $r|p-1, r|q-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \leq i \leq r-1)$.

Suppose that G is a finite group, $Irr(G)$ denotes the set of all irreducible characters of G, $Cl(G)$ is the set of conjugacy classes of G, and $x, y \in G$. Also assume that A is the set of all character values of G, $\mathbb{Q}(A)$ denotes the field generated by $\mathbb{Q}$ and A. If Γ is the Galois group of this extension, then Γ acts on $Irr(G)$ by $\chi^\alpha(g) = \alpha(\chi(g))$. It is well-known that there exists ε such that $\mathbb{Q}(A) \subseteq \mathbb{Q}(\varepsilon)$, where ϵ is a primitive $n-th$ root of unity. Thus if $\alpha \in \Gamma$, then there exists a unique positive integer r such that $(r, n) = 1$ and $\alpha(\varepsilon) = \varepsilon^r$. Therefore, it is well-defined to use the notation $\alpha = \sigma_r$.

The elements x and y are said to be rational conjugates, if $\langle x\rangle$ and $\langle y\rangle$ are the conjugate subgroups of G. The orbits of G under this action are called the rational conjugacy classes of G. On the other hand, Γ acts on the conjugacy classes of G by $(x^G)^{\sigma_r} = (x^r)^G$. It is well known that [4, Corollary 6.33] the number of orbits in the actions of Γ on the irreducible characters and conjugacy classes of G are equal. Moreover, the orbits of Γ on the conjugacy classes of G are the rational conjugacy classes of G.

Lemma 1.1. *Suppose that G is a finite group, A is the set of all character values of G, and $\Gamma = Gal(\frac{\mathbb{Q}(\mathbb{A})}{\mathbb{Q}})$. Then the following hold:*

(1) *If O is an orbit of Γ under its action on conjugacy classes of G, then the union of elements of O is a rational conjugacy class of G, and each rational conjugacy class of G can be obtained in this way.*
(2) *If $O(\chi)$ denotes the orbit of χ under the action of Γ on $Irr(G)$, then the summation of all irreducible characters of $O(\chi)$ is a rational-valued character of G. Moreover, for any proper subset T of $O(\chi)$, the summation of all irreducible characters of T is not rational-valued.*

Proof. Suppose that $A = \{\chi(g) \mid \chi \in Irr(G)\ \&\ g \in G\}$. Then:

(1) Clearly, for each one of the conjugacy class x^G and y^G of an orbit O, there exists $\sigma_r \in \Gamma$ such that $(r, |G|) = 1$ and $\sigma_r(x^G) = (x^r)^G = y^G$. Therefore, there exists $g \in G$ such that $y = g^{-1}x^rg$. This implies that x and y are rational conjugates.
(2) The sum over $O(\chi)$ under the Galois group of the field $\mathbb{Q}(A)$, say, which is generated by the values of χ, is rational because the value of the orbit sum at the group element g is equal to

the trace of $\chi(g)$ with respect to the field extension $\mathbb{Q}(A)/\mathbb{Q}$ multiplied by the order of the stabilizer of $\chi(g)$ in $Gal(\mathbb{Q}(A)/\mathbb{Q})$.

Suppose that the sum over a proper subset T of $O(\chi)$ would be rational-valued, and assume, without loss of generality, that χ is in T. Take an element α in $Gal(\mathbb{Q}(A)/\mathbb{Q})$ that maps χ to a character in the complement of T in $O(\chi)$. Then the sum over the set $\{\psi^{\alpha} \mid \psi \in T\}$ is equal to the sum over T. This contradicts the linear independence of the characters in $O(\chi)$.

This ends the proof. □

Suppose that $\mathcal{X} = \{X_1, \cdots, X_r\}$ denotes the set of orbits of Γ on $Irr(G)$, and $\mathcal{K} = \{K_1, \cdots, K_r\}$ is the set of orbits of Γ on $Cl(G)$. Then the characters $\gamma_i = m_i \sum_{\psi \in X_i} \psi$ and $1 \leq i \leq r$ are called the irreducible rational characters of G, where m_i denotes the Schur index of a character in X_i.

Definition 1.2. Let the pair $(\mathcal{K}, \mathcal{X})$ be as defined above. The square matrix $\mathbb{Q}CT(G) = [a_{ij}]$ with $a_{ij} = \gamma_i(K_j)$ is said to be the rational character table of G, where $\gamma_i(K_j)$ is equal to $\gamma_i(g)$ for any $g \in K_j$.

Throughout this paper, our notations are standard, and can be taken from [2]. Our calculations are carried out with the aid of the computer algebra system GAP [6].

2. Main results

Suppose that G is a group, and A is the set of all character values of G. In the first section, it is shown that there are two actions of $\Gamma = Gal(\frac{\mathbb{Q}(A)}{\mathbb{Q}})$ on $Irr(G)$ and $Cl(G)$. Suppose that G is a finite group with $\mathbb{Q}CT(G) = (\gamma_i(K_j))$, where $1 \leq i, j \leq r$, and r denotes the number of irreducible rational characters. Then the row and column orthogonality relations for the rational character table of G can be written in the following form:

$$\begin{aligned} \langle \gamma_i, \gamma_j \rangle &= \frac{1}{|G|} \sum_{t=1}^{r} |K_t| \gamma_i(K_t) \gamma_j(K_t) = \delta_{i,j} m_i^2 |K_i|, \\ \langle K_p, K_q \rangle &= \sum_{i=1}^{r} \gamma_i(K_p) \gamma_i(K_q) = \delta_{p,q} m_p^2 |C_G(K_p)||K_p|. \end{aligned}$$

Thus if g and g' are two rational conjugate elements, then $|C_G(g)| = |C_G(g')|$. In the above relations, $|K_i|$ is equal to the number of conjugacy classes g^G such that $K_i = \cup g^G$.

The following proposition extends a well-known result of the ordinary character theory to that for the rational character theory [5, Page 167].

Proposition 2.1. *Let G be a finite group. Then,*

$$det(\mathbb{Q}CT(G))^2 = \prod_{i=1}^{r}(|C_G(K_i)| \times |K_i|).$$

Proof. Suppose $Q = \mathbb{Q}CT(G) = (\gamma_i(K_j))$. Then,

$$\begin{aligned} Q^tQ &= (\gamma_j(K_i)) \times (\gamma_i(K_j)) \\ &= (\sum_{i=1}^{r} \gamma_i(K_s)\gamma_i(K_t)) \\ &= diag(|C_G(K_i)| \times |K_i|). \end{aligned}$$

Hence, $det(\mathbb{Q}CT(G))^2 = \prod_{i=1}^{r}(|C_G(K_i)| \times |K_i|)$, proving the result. □

Proposition 2.2. *If G is cyclic, then $\frac{\mathbb{Q}CT(G)^2}{|G|} = I$, where I denotes the identity matrix.*

Proof. Suppose $(\mathcal{K}, \mathcal{X})$ is as definition 1 for G such that $\mathcal{K} = \{K_1, \cdots, K_r\}$. Consider the following two matrices

$$A = diag(\sqrt{|K_1|}, \cdots, \sqrt{|K_r|}),$$

and

$$B = \frac{1}{\sqrt{|G|}} diag(\frac{1}{\sqrt{|K_1|}}, \cdots, \frac{1}{\sqrt{|K_r|}}).$$

Set $C = BQA$, where $Q = \mathbb{Q}CT(G)$. Then $CC^t = I$, and since G is cyclic, $CC^t = \frac{Q^2}{|G|} = I$, as desired. □

Let $\mu(n)$, $\phi(n)$, and $\tau(n)$ denote the Möbius μ-function, Euler totient function, and number of divisor of n, respectively. For $n, m \in \mathbb{Z}$ with $n \geq 1$, the Ramanujan sum $c_n(m)$ is defined as

$$c_n(m) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} e^{\frac{2\pi ikm}{n}}.$$

The Ramanujan sum is always an integer, and the von Sternecks Formula says that $c_n(m) = \frac{\mu\left(\frac{n}{(n,m)}\right)\phi(n)}{\phi\left(\frac{n}{(n,m)}\right)}$.

Theorem 2.3. *For a positive integer n, let d_1, d_2, $\ldots$, $d_{\tau(n)}$ be the divisors of n. The rational character table of the cyclic groups $\mathbb{Z}_n$ can be computed using $\mathbb{Q}CT(\mathbb{Z}_n) = [a_{ij}]$, where $a_{ij} = c_{d_i}\left(\frac{n}{d_j}\right)$.*

Proof. Suppose that $\tau(n)$ denotes the number of divisors of n. The irreducible characters of the cyclic group $\mathbb{Z}_n$ can be computed by $\chi_k(m) = \varepsilon^{km}$, where $\varepsilon = e^{\frac{2\pi i}{n}}$, and the Schur index of each character is equal to 1. It is well-known that $\mathbb{Q}(A) \cong \mathbb{Q}(\varepsilon)$, and $\Gamma = Gal(\frac{\mathbb{Q}(A)}{\mathbb{Q}}) \cong \mathbb{Z}_n^*$. Then $\mathcal{X} = \{X_1, \cdots, X_{\tau(n)}\}$ and $\mathcal{K} = \{K_1, \cdots, K_{\tau(n)}\}$, where $X_i = \{\chi_k \mid (k, n) = \frac{n}{d_i}\}$ and $K_j = \{a \in \mathbb{Z}_n \mid (a, n) = \frac{n}{d_j}\}$. This computes the rational character table of $\mathbb{Z}_n$, as desired. □

Example 2.4. Let p be a prime number. The rational character table of $\mathbb{Z}_p$ is recorded in Table 1. In this case, $K_1 = id_{\mathbb{Z}_p}$ and $K_2 = \mathbb{Z}_p - id_{\mathbb{Z}_p}$.

TABLE 1. Rational Character Table of $\mathbb{Z}_p$.

$\mathbb{Q}CT(\mathbb{Z}_p)$	K_1	K_2
γ_1	1	1
γ_2	$p-1$	-1

Example 2.5. Following James and Liebeck [5, p. 178], the dicyclic group T_{4n} can be presented as follows:

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

It is easy to see that this group has the order $4n$ and the cyclic subgroup $\langle a \rangle$ has the index 2 in T_{4n}. Suppose that $d_1, \cdots, d_{\tau(n)}$ are the positive divisors of n. Let $\varepsilon = e^{\frac{2\pi i}{2n}}$ and $Irr(\langle a \rangle) = \{\psi_1, \psi_2, \cdots, \psi_{2n}\}$, where $\psi_j(a^r) = \varepsilon^{jr}$, and $r = 1, 2, \cdots, 2n$. In [5, p. 420], the character table of T_{4n} is computed. For the sake of completeness, we describe the character table of T_{4n} as follow:

- The group T_{4n} has exactly $n+3$ conjugacy classes as follows: $\{e\}$, $\{a^n\}$, $\{a^r, a^{-r}\}$ $(1 \leq r \leq n-1)$, $\{a^{2j}b \mid 0 \leq j \leq n-1\}$, and $\{a^{2j+1}b \mid 0 \leq j \leq n-1\}$.
- The group T_{4n} has exactly $n-1$ non-linear characters $\bar{\psi}_j$ such that $\bar{\psi}_j \downarrow_{\langle a \rangle} = \psi_j + \psi_{n-j}$, where $1 \leq j \leq n-1$. It is easy to see that $\bar{\psi}_j(a^\alpha b^\beta) = (1-\beta)(\varepsilon^{j\alpha} + \varepsilon^{-j\alpha})$. The Schur index of $\bar{\psi}_j$ is equal to 2, where j is odd and is equal to 1 otherwise.
- The group T_{4n} has exactly four linear characters. These are $\{\overline{\psi}_n, \overline{\psi}'_n\}$ and $\{\chi_1, \chi_2\}$ such that $\overline{\psi}_n$ and $\overline{\psi}'_n$ are two linear characters of T_{4n} with this property that $\overline{\psi}_n \downarrow_{\langle a \rangle} = \overline{\psi}'_n \downarrow_{\langle a \rangle} = \psi_n$, $\chi_1 = 1$ and χ_2 is the lift of the non-trivial irreducible character of $\frac{T_{4n}}{\langle a \rangle} \cong Z_2$ to T_{4n}. Notice that if n is odd, then

$\overline{\psi}_n(a^\alpha b^\beta) = (-1)^\alpha i^\beta$ and $\overline{\psi}'_n(a^\alpha b^\beta) = (-1)^\alpha(-i)^\beta$. If n is even, $\overline{\psi}_n(a^\alpha b^\beta) = (-1)^\alpha(-1)^\beta$ and $\overline{\psi}'_n(a^\alpha b^\beta) = (-1)^\beta$. It is obvious the Schur index of these characters is equal to 1.

On the other hand, $\mathbb{Q}(A) = \mathbb{Q}(\varepsilon + \varepsilon^{-1})$, where A is the character values of T_{4n} and $[\mathbb{Q}(A) : \mathbb{Q}] = \frac{\phi(2n)}{2}$. Thus the Galois group is abelian of order $\frac{\phi(2n)}{2}$. Suppose that n is odd. Then the action of Galois group Γ on $Irr(T_{4n})$ has exactly $\tau(2n)+1$ orbits, as follow:

$$\begin{aligned}
X_1 &= \{1\},\\
X_2 &= \{\chi_2\},\\
X_i &= \{\overline{\psi_j} \mid (j, 2n) = \frac{2n}{d_i}; j = 1, \cdots, n-1\} \ (3 \leq i \leq \tau(2n)),\\
X_{\tau(2n)+1} &= \{\overline{\psi}_n, \overline{\psi'}_n\}.
\end{aligned}$$

Notice that $|X_i| = \frac{\phi(d_i)}{2}$. We now consider the action of Galois group Γ on $Cl(T_{4n})$. The orbits of this action are as follows:

$$\begin{aligned}
K_1 &= \{1\},\\
K_2 &= \{a^n\},\\
K_i &= \{a^r, a^{2n-r} \mid (r, 2n) = \frac{2n}{d_i}\} \ (3 \leq i \leq \tau(2n)),\\
K_{\tau(2n)+1} &= \{a^j b \mid 0 \leq j \leq 2n-1\}.
\end{aligned}$$

Then $\mathcal{X} = \{X_1, X_2, \cdots, X_{\tau(2n)}, X_{\tau(2n)+1}\}$ and $\mathcal{K} = \{K_1, K_2, \cdots, K_{\tau(2n)}, K_{\tau(2n)+1}\}$. We now assume that n is even. Then, similar to the case of odd n, we have exactly $\tau(2n)+2$ orbits, as follow:

$$\begin{aligned}
X_1 &= \{1\},\\
X_2 &= \{\chi_2\},\\
X_i &= \{\overline{\psi_j} \mid (j, 2n) = \frac{2n}{d_i}; j = 1, \cdots, n-1\} \ (3 \leq i \leq \tau(2n)),\\
X_{\tau(2n)+1} &= \{\overline{\psi}_n\},\\
X_{\tau(2n)+2} &= \{\overline{\psi'}_n\}.
\end{aligned}$$

Again $|X_i| = \frac{\phi(d_i)}{2}$, and we have:

$$\begin{array}{rcl} K_1 &=& \{1\}, \\ K_2 &=& \{a^n\}, \\ K_i &=& \{a^r, a^{2n-r} \mid (r, 2n) = \dfrac{2n}{d_i}\} \ (3 \leq i \leq \tau(2n)), \\ K_{\tau(2n)+1} &=& \{a^{2j}b \mid 0 \leq j \leq n-1\}, \\ K_{\tau(2n)+2} &=& \{a^{2j+1}b \mid 0 \leq j \leq n-1\}. \end{array}$$

Therefore, we have:

$$\begin{array}{rcl} X &=& \{X_1, X_2, \cdots, X_{\tau(2n)+1}, X_{\tau(2n)+2}\}, \\ \mathcal{K} &=& \{K_1, K_2, \cdots, K_{\tau(2n)+1}, K_{\tau(2n)+2}\}. \end{array}$$

Notice that, by [7], the Schur index of T_{4n} can be computed as follows:

$$t_i = \begin{cases} 1 & X_i \text{ contains } \bar{\psi}_j \text{ when } j \text{ is even} \\ 2 & X_i \text{ contains } \bar{\psi}_j \text{ when } j \text{ is odd} \end{cases}.$$

The rational character tables of T_{4n} are recorded in Tables 2 and 3.

Table 2. Rational Character Table of T_{4n}; n is odd.

$\mathbb{Q}CT(T_{4n})$	K_1	K_2	K_i	$K_{\tau(2n)+1}$
γ_1	1	1	1	1
γ_2	1	1	1	-1
γ_i	$t_i\phi(d_i)$	$t_ic_{d_i}(n)$	$t_ic_{d_i}(r)$	0
$\gamma_{\tau(2n)+1}$	2	2	$2 \times (-1)^r$	0

Table 3. Rational Character Table of T_{4n}, n is even.

$\mathbb{Q}CT(T_{4n})$	K_1	K_2	K_i	$K_{\tau(2n)+1}$	$K_{\tau(2n)+2}$
γ_1	1	1	1	1	1
γ_2	1	1	1	-1	-1
γ_i	$t_i\phi(d_i)$	$t_ic_{d_i}(n)$	$t_ic_{d_i}(r)$	0	0
$\gamma_{\tau(2n)+1}$	1	1	$(-1)^r$	-1	1
$\gamma_{\tau(2n)+2}$	1	1	$(-1)^r$	1	-1

Lemma 2.6. *Let A, A_1, and A_2 be the set of all entries of the character tables $G_1 \times G_2$, G_1 and G_2, respectively. Let $\Gamma_1 = Gal(\frac{\mathbb{Q}(A_1)}{\mathbb{Q}})$, $\Gamma_2 = Gal(\frac{\mathbb{Q}(A_2)}{\mathbb{Q}})$ and $\Gamma = Gal(\frac{\mathbb{Q}(A)}{\mathbb{Q}})$. If $gcd(|G_1|, |G_2|) = 1$, then $\Gamma = \Gamma_1 \times \Gamma_2$.*

Proof. The proof follows from the fact that all entries of the character tables G_1 and G_2 are the sum of roots of unity of the orders $|G_1|$ and $|G_2|$, respectively. $\square$

Lemma 2.7. [4, Exercise 10.10] *Let $\chi \in Irr(G_1)$, $\psi \in Irr(G_2)$, and $gcd(m(\chi), \psi(1)|\mathbb{Q}(\psi) : \mathbb{Q}|) = gcd(m(\psi), \chi(1)|\mathbb{Q}(\chi) : \mathbb{Q}|) = 1$. Then $m(\chi\psi) = m(\chi)m(\psi)$.*

Theorem 2.8. *Suppose that G_1 and G_2 are finite groups of the orders n_1 and n_2, respectively, where $gcd(n_1, n_2) = 1$. If χ and ψ satisfy the hypothesis of Lemma 2.7 for any $\chi \in Irr(G_1)$ and $\psi \in Irr(G_2)$, then we have*

$$\mathbb{Q}CT(G_1 \times G_2) = \mathbb{Q}CT(G_1) \otimes \mathbb{Q}CT(G_2).$$

Proof. By Lemma 2.6, $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_1 = Gal(\frac{\mathbb{Q}(A_1)}{\mathbb{Q}})$, $\Gamma_2 = Gal(\frac{\mathbb{Q}(A_2)}{\mathbb{Q}})$, and $\Gamma = Gal(\frac{\mathbb{Q}(A)}{\mathbb{Q}})$. Let A, A_1 and A_2 be the entries of the character tables $G_1 \times G_2$, G_1, and G_2, respectively. Suppose that the action of Γ_1 on $Irr(G_1)$ and the action of Γ_2 on $Irr(G_2)$ have the classes $\{X_1, \ldots, X_r\}$ and $\{Y_1, \ldots, Y_s\}$, respectively. It is well-known that, $Irr(G_1 \times G_2) = Irr(G_1) \times Irr(G_2)$. Thus it is sufficient to prove that for each $\chi \in X_i$, $(1 \leq i \leq r)$ and for each $\psi \in Y_j$, $(1 \leq j \leq s)$, $[\chi\psi] = [\chi][\psi]$, where $[\chi] = \displaystyle\sum_{\sigma \in \Gamma} \chi^\sigma$. It is clear that for every $\sigma \in \Gamma$, there exist $\sigma_1 \in \Gamma_1$ and $\sigma_2 \in \Gamma_2$ such that $\sigma = (\sigma_1, \sigma_2)$. This implies that

$$\begin{aligned} [\chi\psi] &= \{(\chi\psi)^\sigma | \sigma \in \Gamma\} \\ &= \{(\chi\psi)^{(\sigma_1, \sigma_2)} | \sigma_1 \in \Gamma_1, \sigma_2 \in \Gamma_2\} \\ &= \{\chi^{\sigma_1}\psi^{\sigma_2} | \sigma_1 \in \Gamma_1, \sigma_2 \in \Gamma_2\} = [\chi][\psi]. \end{aligned}$$

Suppose that $X_i = \{\chi_{i_t}\}_t$ and $Y_j = \{\psi_{j_l}\}_l$, then if $\gamma_i = m_i \sum_t \chi_{i_t}$ and $\delta_j = m_j \sum_l \psi_{j_l}$, by Lemma 2.7, we have:

$$\gamma_i \delta_j = m_i m_j \sum_{t,l} \chi_{i_t} \psi_{j_l}.$$

Similarly, the rational conjugacy classes of $G_1 \times G_2$ are equal to the product of the rational conjugacy classes of G_1 and G_2. Let K_f and L_t be the rational conjugacy classes of G_1 and G_2, respectively. Then we have

$$\gamma_{\chi_i \psi_j}((K_f, L_s)) = \gamma_{\chi_i}(K_f)\gamma_{\psi_j}(L_s).$$

This completes the proof. $\square$

Example 2.9. Let p and q be two prime numbers such that $p > q$, and G be a finite group of the order pq. It is well-known that $G \cong \mathbb{Z}_{pq}$ or $G \cong T_{p,q}$. As a result of Theorem 2.8, one can easily prove that the rational character table of $\mathbb{Z}_p \times \mathbb{Z}_q$ is $\mathbb{Q}CT(\mathbb{Z}_p) \otimes \mathbb{Q}CT(\mathbb{Z}_q)$.

TABLE 4. Rational Character Table of $\mathbb{Z}_{pq}$.

$\mathbb{Q}CT(\mathbb{Z}_{pq})$	K_1	K_2	K_3	K_4
γ_1	1	1	1	1
γ_2	$q-1$	-1	$q-1$	-1
γ_3	$p-1$	$p-1$	-1	-1
γ_4	$pq-p-q+1$	$1-p$	$1-q$	1

Here, $K_1 = \{id\}$, $K_2 = \{p, 2p, \ldots, (q-1)p\}$, $K_3 = \{q, 2q, \ldots, (p-1)q\}$, and $K_4 = a \in \mathbb{Z}_{pq} : (a, pq) = 1\}$.

Proposition 2.10. *Suppose that p is prime, and $q \mid p-1$ and $d_1, \cdots, d_{\tau(q)}$ are the positive divisors of q. Then the rational character table of $G = F_{p,q}$ is given in Table 5.*

TABLE 5. Rational Character Table of $F_{p,q}$.

$\mathbb{Q}CT(F_{p,q})$	K_1	K_2	K_j
γ_1	1	1	1
γ_i	$\phi(d_i)$	$\phi(d_i)$	$c_{d_i}\left(\frac{q}{d_j}\right)$
$\gamma_{\tau(q)+1}$	$p-1$	-1	0

Proof. Suppose that $\alpha = e^{\frac{2\pi i}{q}}$ and $\beta = e^{\frac{2\pi i}{p}}$. According to [5, Proposition 25.9 and Theorem 25.10], the q linear characters of $F_{p,q}$ are defined as $\chi_i(a^x b^y) = \alpha^{iy}$, where $0 \leq i \leq q-1$. Moreover, the $r = \frac{p-1}{q}$ non-linear characters of $F_{p,q}$ are

$$\psi_j(g) = \begin{cases} \sum_{s \in S} \beta^{sxv_j} & g = a^x \\ 0 & g = a^x b^y \end{cases},$$

where S is a subgroup of $\mathbb{Z}_p^\star$ containing all powers of u, and the representatives of the cosets of S in $\mathbb{Z}_p^\star$ can be considered as $\{v_1, v_2, \ldots, v_r\}$. The conjugacy classes of $F_{p,q}$ are:

$$\begin{aligned} &\{1\}, \\ &(a^{v_i})^G = \{a^{v_i s} : s \in S\}, \quad (1 \leq i \leq r), \\ &(b^n)^G = \{a^m b^n : 0 \leq m \leq p-1\}, \quad (1 \leq n \leq q-1). \end{aligned}$$

Then the action of the Galois group $\Gamma = Gal(\frac{\mathbb{Q}(\alpha,\sum_{s\in S}\beta^s)}{\mathbb{Q}})$ on $Irr(F_{p,q})$ has exactly $\tau(q)+1$ orbits, as:

$$\begin{aligned} X_1 &= \{1\}, \\ X_i &= \{\chi_j \mid (j,q) = \frac{q}{d_i}; j = 1, \cdots, \tau(q)\} \ (2 \le i \le \tau(q)), \\ X_{\tau(q)+1} &= \{\psi_1, \psi_2, \ldots, \psi_r\}. \end{aligned}$$

On the other hand, the orbits of the action of Galois group Γ on $Cl(F_{p,q})$ are:

$$\begin{aligned} K_1 &= \{1\}, \\ K_2 &= \{a^i : 1 \le i \le p-1\}, \\ K_{j+1} &= \cup_{(k,q)=\frac{q}{d_j}}(b^k)^G, \ (2 \le j \le \tau(q)). \end{aligned}$$

By [1, Theorem 4.2], the Schur index of all irreducible characters of $F_{p,q}$ is 1, and the proof is completed. □

Corollary 2.11. *Let p and q be two prime numbers such that $p > q$ and $q|p-1$. The rational character table of the group $T_{p,q}$ is as Table 6.*

In this case, the rational conjugacy classes are $K_1 = \{id\}$, $K_2 = \langle a\rangle \setminus \{id\}$ and $K_3 = T_{p,q} \setminus \langle a\rangle$.

TABLE 6. Rational Character Table of $T_{p,q}$.

$\mathbb{Q}CT(T_{p,q})$	K_1	K_2	K_3
γ_1	1	1	1
γ_2	$q-1$	$q-1$	-1
γ_3	$p-1$	-1	0

From now on, we consider the groups of the order pqr $(p > q > r)$. To compute the rational character tables of a group G of this order, we first calculate the character table of all groups of this order. In Theorem 2.3, the rational character table of the cyclic $\mathbb{Z}_n$ is computed. Thus it is enough to assume that group G is non-cyclic. We now apply Theorems 2.3 and 2.8 to compute the rational character tables of $\mathbb{Z}_r \times F_{p,q}$, $\mathbb{Z}_q \times T_{p,r}$ and $\mathbb{Z}_p \times T_{q,r}$ as $\mathbb{Q}CT(\mathbb{Z}_r) \otimes \mathbb{Q}CT(F_{p,q})$, $\mathbb{Q}CT(\mathbb{Z}_q) \otimes \mathbb{Q}CT(T_{p,r})$, and $\mathbb{Q}CT(\mathbb{Z}_p) \otimes \mathbb{Q}CT(T_{q,r})$, respectively.

The character table of G_5 can be computed directly from Proposition 2.10. We now compute the character table of groups G_{i+5}. Let $G =$

G_{i+5}, to compute the conjugacy classes of G. We assume that $U = \langle u \rangle$ and $V = \langle v \rangle$ are the subgroups of order r of $\mathbb{Z}_q^*$ and $\mathbb{Z}_p^*$, respectively. Then we have,

Lemma 2.12. *The conjugacy classes of G are*

$$\{1\}, (a^{v_i})^G, (b^{u_i})^G, (c^i)^G, (b^{u_i'}a^{v_i'})^G$$

where u_i is a coset representative of U in $\mathbb{Z}_q^*$, v_i is a coset representative of V in $\mathbb{Z}_p^*$, and (u_i', v_i') is a coset representative of $\langle (u, v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$.

Proof. It is easy to see that for $1 \leq k \leq r-1$, $c^{-k}bc^k = b^{u^k}$ and so b^{u^i}'s are conjugate and so $|b^G| \geq r$. On the other hand, $\langle ba \rangle \leq C_G(b)$, and hence, $|C_G(b)| \geq pq$. This implies that $|b^G| \leq r$, and thus $|b^G| = r$. Further, one can prove that $(b^{u_i})^G (1 \leq i \leq \frac{q-1}{r})$ and $(a^{v_j})^G (1 \leq j \leq \frac{p-1}{r})$ are conjugacy classes of G. Therefore,

$$\begin{aligned}
c^G &= \{cb^i a^j \mid 0 \leq i \leq q-1, 0 \leq j \leq p-1\}, \\
&\vdots \\
(c^{r-1})^G &= \{c^{r-1} b^i a^j \mid 0 \leq i \leq q-1, 0 \leq j \leq p-1\}, \\
(b^{u_i'} a^{v_i'})^G &= \{b^{u_i'} a^{v_i'}, b^{u_i' u} a^{v_i' v}, \cdots, b^{u_i' u^{r-1}} a^{v_i' v^{r-1}}\},
\end{aligned}$$

where (u_i', v_i') is a coset representative of $\langle (u, v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$ and $|\langle (u, v) \rangle| = r$. $\square$

It follows from Lemma 2.12 that G has $\frac{p-1}{r} + \frac{q-1}{r} + \frac{(p-1)(q-1)}{r} + r$ conjugacy classes and then the same number of irreducible characters. On the other hand,

$$G/G' = \langle c | c^r = 1 \rangle \cong \mathbb{Z}_r.$$

Hence, G has r linear characters lifted from linear characters of G/G'. These characters are as $\tilde{\chi}_n : G/G' \to C$ with $\tilde{\chi}_n(c^m G') = \epsilon^{mn}$, where $\epsilon = e^{\frac{2\pi i}{r}}$ and $m, n = 0, 1, \cdots, r-1$.

According to [5, Theorem 17.11], all linear characters of G are as $\chi_n : G \to C$ with $\chi_n(g) = \tilde{\chi}_n(gG')$. Hence,

$$\begin{aligned}
\chi_n(a^w) &= \tilde{\chi}_n(a^w G') = \tilde{\chi}_n(G') = \chi_n(1) = 1, \\
\chi_n(b^v) &= \tilde{\chi}_n(b^v G') = \tilde{\chi}_n(G') = \chi_n(1) = 1, \\
\chi_n(b^{v_0} a^{w_0}) &= \tilde{\chi}_n(b^{v_0} a^{w_0} G') = \tilde{\chi}_n(G') = \chi_n(1) = 1, \\
\chi_n(c^t) &= \tilde{\chi}_n(c^t G') = \epsilon^{tn} (0 \leq n \leq r-1 \textit{ and } 1 \leq t \leq r-1).
\end{aligned}$$

where (v_0, w_0) is a coset representative of $\langle (u, v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$.

Here, we determine all the non-linear irreducible characters of G. First notice that $H = \langle a \rangle$ is a normal subgroup of G, and if $u^r \equiv 1(mod\ q)$, then

$$G/H = \langle b, c |\ b^q = c^r = 1, c^{-1}bc = b^u \rangle \cong F_{q,r}.$$

According to Proposition 2.10, the Frobenius group $F_{q,r}$ has r linear characters and $\frac{q-1}{r}$ irreducible characters of degree r. Let us denote the non-linear characters by $\tilde{\varphi}_m$. Then we have:

$$\begin{aligned}
\tilde{\varphi}_m(H) &= r, \\
\tilde{\varphi}_m(b^x H) &= \sum_{i=0}^{r-1} \lambda^{u_m x u^i} (1 \le m \le \frac{q-1}{r}, 1 \le x \le q-1), \\
\tilde{\varphi}_m(b^x c^y H) &= 0\ (1 \le y \le r-1),
\end{aligned}$$

where $\lambda = e^{\frac{2\pi i}{q}}$ and $u_1, \cdots, u_m$ are distinct coset representative of $U = \langle u \rangle$ in $\mathbb{Z}_q^*$. By lifting these characters, we can compute $\frac{q-1}{r}$ irreducible characters of G of degree r denoted by $\varphi_m(1 \le m \le \frac{q-1}{r})$, e.g.

$$\begin{aligned}
\varphi_m(a^x) &= r\ (0 \le x \le p-1), \\
\varphi_m(b^y a^x) &= \sum_{i=0}^{r-1} \lambda^{u_m y u^i} (0 \le x \le p-1, 1 \le y \le q-1), \\
\varphi_m(c^k) &= 0\ (1 \le k \le r-1).
\end{aligned}$$

Similarly, for the normal subgroup $K = \langle b \rangle$ of G, we have:

$$G/K \cong \langle a, c |\ a^p = c^r = 1, c^{-1}ac = a^v \rangle \cong F_{p,r}.$$

Consequently, this group has $\frac{p-1}{r}$ irreducible characters of degree r denoted by $\tilde{\theta}_l(1 \le l \le \frac{p-1}{r})$. Similar to the last discussion, the irreducible characters of G lifted from $\tilde{\theta}_l$ are as follows:

$$\begin{aligned}
\theta_l(a^x) &= \theta_l(b^y a^x) = \sum_{i=0}^{r-1} \gamma^{v_l x v^i}, \\
\theta_l(b^y) &= r, \\
\theta_l(c^k) &= 0\ (1 \le k \le r-1),
\end{aligned}$$

where $\gamma = e^{\frac{2\pi i}{p}}$ and $v_1, \cdots, v_l$ are distinct coset representative of $V = \langle v \rangle$ in $\mathbb{Z}_p^*$.

Finally, by considering subgroup $G' = \langle ba \rangle \cong \mathbb{Z}_q \times \mathbb{Z}_p$, its irreducible characters are of the form $\psi_i \xi_j (0 \le i \le q-1, 0 \le j \le p-1)$ and

$$\psi_i(b^y) = \lambda^{iy}, \xi_j(a^x) = \gamma^{jx}.$$

This leads us to conclude that

$$\psi_i\xi_j(b^y a^x) = \psi_i(b^y)\xi_j(a^x) = \lambda^{iy}\gamma^{jx}.$$

Let now $m \in \mathbb{Z}_q^*$ and $n \in \mathbb{Z}_p^*$, then

$$(\psi_m\xi_n \uparrow G)(1) = \frac{|G|}{|\langle ba\rangle|}(\psi_m\xi_n)(1) = \frac{pqr}{pq} = r.$$

On the other hand,

$$\begin{aligned} |C_G(b^y)| &= |C_{G'}(b^y)| = |C_G(a^x)| = |C_{G'}(a^x)| = |C_G(b^y a^x)| \\ &= |C_{G'}(b^y a^x)| = pq, \end{aligned}$$

and so

$$\begin{aligned} (\psi_m\xi_n \uparrow G)(a^x) &= \sum_{i=0}^{r-1} \xi_n(a^{xv^i}) = \sum_{i=0}^{r-1} \gamma^{nxv^i}, \\ (\psi_m\xi_n \uparrow G)(b^y) &= \sum_{i=0}^{r-1} \psi_m(b^{yu^i}) = \sum_{i=0}^{r-1} \lambda^{myu^i}, \\ (\psi_m\xi_n \uparrow G)(b^y a^x) &= \sum_{i=0}^{r-1} \psi_m(b^{yu^i})\xi_n(a^{xv^i}) = \sum_{i=0}^{r-1} \lambda^{myu^i}\gamma^{nxv^i} \\ (\psi_m\xi_n \uparrow G)(c^k) &= 0 \ (k = 1, \cdots, r-1). \end{aligned}$$

Since

$$\psi_m\xi_n \uparrow G = \psi_{mu^i}\xi_{nv^i} \uparrow G,$$

thus, we get $z = \frac{(p-1)(q-1)}{r}$ characters of G. There still remains the question as to whether such characters are distinct irreducible. Assume (u_i', v_i') be coset representative of subgroup $\{(1,1), (u,v), \cdots, (u^{r-1}, v^{r-1})\}$ of $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$ and $\eta_j = \psi_{u_j'}\xi_{v_j'} \uparrow G$. According to Frobenius Reciprocity Theorem, for $H = G' = \langle ba\rangle$, we verify:

$$\begin{aligned} \langle \eta_j \downarrow H, \psi_{u_j' u^i}\xi_{v_j' v^j}\rangle_H &= \langle \eta_j, \psi_{u_j'}\xi_{v_j'} \uparrow G\rangle_G \\ &= \langle \eta_j, \eta_j\rangle_G. \end{aligned}$$

Therefore, we can observe that

$$\eta_j \downarrow H = \langle \eta_j, \eta_j\rangle_G (\sum_{i=0}^{r-1} \psi_{v_j' u^i}\xi_{v_j' v^i}) + \chi,$$

where $\chi = 0$ or it is a character of H. Hence, $\eta_j(1) \geq r\langle \eta_j, \eta_j\rangle_G$. Finally, $\eta_j(1) = r$ implies that $\langle \eta_j, \eta_j\rangle_G = 1$, and so η_j is irreducible. On the other hand, for $(u_j', v_j') \in \mathbb{Z}_q^* \times \mathbb{Z}_p^*$, all $\psi_{u_j'}\xi_{v_j'}$'s are linearly independent, and thus all $\eta_j \downarrow H$ $(1 \leq j \leq \frac{(p-1)(q-1)}{r})$ are distinct. Consequently, the irreducible characters $\eta_1, \cdots \eta_z$ are distinct. We summarize the character table of G in the following theorem:

Theorem 2.13. *Let $p > q > r$ be distinct prime numbers, $l_1 = \frac{(p-1)(q-1)}{r}, l_2 = \frac{p-1}{r}, l_3 = \frac{q-1}{r}$ and $\epsilon = e^{\frac{2\pi i}{r}}$. Then the group G has $l_1 + l_2 + l_3 + r$ irreducible characters, as reported in Table 7.*

TABLE 7. Character Table of Group G_{i+5} for $1 \leq i \leq r$.

g	1	a^{v_i} $1 \leq i \leq l_1$	b^{u_i} $1 \leq i \leq l_2$	$b^{u_i'}a^{v_i'}$ $1 \leq i \leq l_3$	c^k $1 \leq k \leq r-1$
χ_n $0 \leq n \leq r-1$	1	1	1	1	ϵ^{kn}
η_s $1 \leq s \leq l_3$	r	E	F	G	0
θ_l $1 \leq l \leq l_1$	r	C	r	D	0
φ_m $1 \leq m \leq l_2$	r	r	A	B	0

where $\lambda = e^{\frac{2\pi i}{q}}$, $u_1, \cdots, u_{l_1}$ are distinct coset representative of $U = \langle u \rangle$ in $\mathbb{Z}_q^*$, $v_1, \cdots, v_{l_2}$ are distinct coset representative of $V = \langle v \rangle$ in $\mathbb{Z}_p^*$, (u_i', v_i') are coset representative of $\langle (u, v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$ and

$$A = \sum_{j=1}^{r} \lambda^{u_m u_i u^j}, B = \sum_{j=1}^{r} \lambda^{u_m u_i' u^j}, C = \sum_{j=1}^{r} \gamma^{v_l v_i v^j}, D = \sum_{j=1}^{r} \gamma^{v_l v_i' v^j},$$

$$E = \sum_{j=1}^{r} \gamma^{v_s' v_i v^j}, F = \sum_{j=1}^{r} \lambda^{u_s' u_i u^j}, G = \sum_{j=1}^{r} \lambda^{u_s' u_i' u^j} \gamma^{v_s' v_i' v^j}.$$

and $1 \leq l \leq l_1$, $1 \leq m \leq l_2$, $1 \leq s \leq l_3$, $1 \leq n \leq r-1$.

Suppose $\mathfrak{G}(p, q, r)$ be the set of all groups of order pqr, where p, q, and r are distinct prime numbers with $p > q > r$.

Theorem 2.14. *Let $G \in \mathfrak{G}(p, q, r)$. Then, the rational character table of G is equal to one of the Tables 8-13.*

Proof. Using Theorems 2.3, 2.8 and Proposition 2.10, the rational character table of groups $G_1 - G_5$ are as reported in Tables 8-12. Let $G = G_{i+5}$ for $1 \leq i \leq r-1$. According to [7, Theorem 3], the Schur index of G is equal to 1. Since $p, q, andr$ are prime numbers, one can easily check that $\langle a^s \rangle = \langle a^t \rangle$, $\langle b^m \rangle = \langle b^n \rangle$, $\langle c^k \rangle = \langle c^l \rangle$, and $\langle b^x a^y \rangle = \langle b^v a^z \rangle$, where $1 \leq s, t, y, z \leq p-1$, $1 \leq m, n, x, v \leq q-1$, $1 \leq k, l \leq r-1$. This implies that all rational conjugacy classes are

$$K_1 = \{id\},\ K_2 = \cup_{i=1}^{l_1}(a^{v_i})^G,\ K_3 = \cup_{i=1}^{l_2}(b^{u_i})^G,$$

$$K_4 = \cup_{i=1}^{l_3}(b^{u'_i}a^{v'_i})^G, K_5 = \cup_{k=1}^{r-1}(c^k)^G.$$

On the other hand, let $\mathbb{Q}(A) = \mathbb{Q}(\epsilon, \lambda^u + \lambda^{u^{-1}}, \gamma^v + \gamma^{v^{-1}})$, so the action of Galois group Γ on $Irr(G)$ has exactly five orbits, as follow:

$$X_1 = \{1\}, X_2 = \{\chi_n : 1 \leq n \leq r-1\}, X_3 = \{\eta_s : 1 \leq s \leq l_3\},$$

$$X_4 = \{\theta_l : 1 \leq l \leq l_1\}, X_4 = \{\varphi_m : 1 \leq m \leq l_2\}.$$

This completes the proof. □

3. Appendix

In this section, the rational character tables of the groups G_1, G_2, G_3, G_4, G_5 and G_{i+5}, $1 \leq i \leq r-1$, all of which having the order pqr are presented. These tables are computed by the methods given by Proposition 2.10 and Theorems 2.3, 2.8, and 2.14.

Table 8. Rational Character Table of Group G_1.

$\mathbb{Q}CT(G_1)$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
γ_1	1	1	1	1	1	1	1	1
γ_2	$r-1$	-1	$r-1$	-1	$r-1$	-1	$r-1$	-1
γ_3	$q-1$	$q-1$	-1	-1	$q-1$	$q-1$	-1	-1
γ_4	a_1	$1-q$	$1-r$	1	a_1	$1-q$	$1-r$	1
γ_5	$p-1$	$p-1$	$p-1$	$p-1$	-1	-1	-1	-1
γ_6	a_2	$1-p$	a_2	$1-p$	$1-r$	1	$1-r$	1
γ_7	a_3	$p-1$	-1	-1	$1-q$	$1-q$	1	1
γ_8	a_4	$-a_3$	$-a_2$	$p-1$	$-a_1$	$1-q$	-1	-1

Here, $a_1 = (q-1)(r-1)$, $a_2 = (p-1)(r-1)$, $a_3 = (p-1)(q-1)$, and $a_4 = (p-1)(q-1)(r-1)$. The rational conjugacy classes are,

$$\begin{aligned}
K_1 &= \{id_{G_1}\},\\
K_2 &= \{pq, 2pq, \ldots, (r-1)pq\},\\
K_3 &= \{pr, 2pr, \ldots, (q-1)pr\},\\
K_4 &= \{qr, 2qr, \ldots, (p-1)qr\},\\
K_5 &= \{p, 2p, \ldots, (r-1)(q-1)p\},\\
K_6 &= \{q, 2q, \ldots, (r-1)(p-1)q\},\\
K_7 &= \{r, 2r, \ldots, (q-1)(p-1)r\},\\
K_8 &= \{a \in \mathbb{Z}_{pqr} : (a, pqr) = 1\}.
\end{aligned}$$

TABLE 9. Rational Character Table of Group G_2.

$\mathbb{Q}CT(G_2)$	K_1	K_2	K_3	K_4	K_5	K_6
γ_1	1	1	1	1	1	1
γ_2	$r-1$	-1	$r-1$	-1	$r-1$	-1
γ_3	$q-1$	$q-1$	$q-1$	$q-1$	-1	-1
γ_4	a_1	$1-q$	a_1	$1-q$	$1-r$	1
γ_5	$p-1$	$p-1$	-1	-1	0	0
γ_6	a_2	$1-p$	$1-r$	1	0	0

Here, $a_1 = (q-1)(r-1)$ and $a_2 = (p-1)(r-1)$. Let $F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u\rangle$. Also consider that $L_1 = \{id_{\mathbb{Z}_r}\}$ and $L_2 = \mathbb{Z}_r \setminus L_1$ are the rational conjugacy classes of $\mathbb{Z}_r$ and $T_1 = \{id_{F_{p,q}}\}$, $T_2 = \{a, \ldots, a^{p-1}\}$ and $T_3 = F_{p,q} \setminus \langle a\rangle$ are the rational conjugacy classes of $F_{p,q}$. Then we have $K_1 = L_1 \times T_1$, $K_2 = L_2 \times T_1$, $K_3 = L_1 \times T_2$, $K_4 = L_2 \times T_2$, $K_5 = L_1 \times T_3$, and $K_6 = L_2 \times T_3$.

TABLE 10. Rational Character Table of Group G_3.

$\mathbb{Q}CT(G_3)$	K_1	K_2	K_3	K_4	K_5	K_6
γ_1	1	1	1	1	1	1
γ_2	$q-1$	-1	$q-1$	-1	$q-1$	-1
γ_3	$r-1$	$r-1$	$r-1$	$r-1$	-1	-1
γ_4	a_1	$1-r$	a_1	$1-r$	$1-q$	1
γ_5	$p-1$	$p-1$	-1	-1	0	0
γ_6	a_2	$1-p$	$1-q$	1	0	0

Here, $a_1 = (q-1)(r-1)$ and $a_2 = (p-1)(q-1)$. Let $F_{p,r} = \langle a, b : a^p = b^r = 1, b^{-1}ab = a^u\rangle$. Consider $L_1 = \{id_{\mathbb{Z}_q}\}$ and $L_2 = \mathbb{Z}_q \setminus L_1$ are the rational conjugacy classes of $\mathbb{Z}_q$ and $T_1 = \{id_{F_{p,r}}\}$, $T_2 = \{a, \ldots, a^{p-1}\}$, and $T_3 = F_{p,r} \setminus \langle a \rangle$ are the rational conjugacy classes of $F_{p,r}$. Then $K_1 = L_1 \times T_1$, $K_2 = L_2 \times T_1$, $K_3 = L_1 \times T_2$, $K_4 = L_2 \times T_2$, $K_5 = L_1 \times T_3$, and $K_6 = L_2 \times T_3$.

TABLE 11. Rational Character Table of Group G_4.

$\mathbb{Q}CT(G_4)$	K_1	K_2	K_3	K_4	K_5	K_6
γ_1	1	1	1	1	1	1
γ_2	$p-1$	-1	$p-1$	-1	$p-1$	-1
γ_3	$r-1$	$r-1$	$r-1$	$r-1$	-1	-1
γ_4	a_1	$1-r$	a_1	$1-r$	$1-p$	1
γ_5	$q-1$	$q-1$	-1	-1	0	0
γ_6	a_2	$1-q$	$1-p$	1	0	0

Here, $a_1 = (q-1)(r-1)$. Let $F_{q,r} = \langle a, b : \ a^q = b^r = 1, b^{-1}ab = a^u \rangle$. Consider $L_1 = \{id_{\mathbb{Z}_p}\}$ and $L_2 = \mathbb{Z}_p \setminus L_1$ are the rational conjugacy classes of $\mathbb{Z}_p$, and $T_1 = \{id_{F_{q,r}}\}$, $T_2 = \{a, \ldots, a^{q-1}\}$ and, $T_3 = F_{q,r} \setminus \langle a \rangle$ are the rational conjugacy classes of $F_{q,r}$. Then $K_1 = L_1 \times T_1$, $K_2 = L_2 \times T_1$, $K_3 = L_1 \times T_2$, $K_4 = L_2 \times T_2$, $K_5 = L_1 \times T_3$, and $K_6 = L_2 \times T_3$.

TABLE 12. Rational Character Table of Group G_5.

$\mathbb{Q}CT(G_5)$	K_1	K_2	K_3	K_4	K_5
γ_1	1	1	1	1	1
γ_2	$r-1$	$r-1$	-1	$r-1$	-1
γ_3	$q-1$	$q-1$	$q-1$	-1	-1
γ_4	a_1	a_1	$1-q$	$1-r$	1
γ_5	$p-1$	-1	0	0	0

Here, $a_1 = (p-1)(r-1)$ and $a_2 = (p-1)(q-1)$. Let $F_{p,qr} = \langle a, b : \ a^p = b^{qr} = 1, b^{-1}ab = a^u \rangle$. Then, the rational conjugacy classes are, $K_1 = \langle id_{G_5} \rangle$, $K_2 = \{a^i : 1 \leq i \leq p-1\}$, $K_3 = \cup_{i=1}^{r-1}(b^{iq})^{G_5}$, $K_4 = \cup_{i=1}^{q-1}(b^{ir})^{G_5}$ and $K_5 = \cup_{(k,qr)=1}(b^k)^{G_5}$.

TABLE 13. Rational Character Table of Group $G = G_{i+5}$, $1 \le i \le r-1$.

$\mathbb{Q}CT(G_{i+5})$	K_1	K_2	K_3	K_4	K_5
γ_1	1	1	1	1	1
γ_2	$r-1$	$r-1$	$r-1$	$r-1$	-1
γ_3	$q-1$	$q-1$	-1	-1	0
γ_4	$p-1$	-1	$p-1$	-1	0
γ_5	$(p-1)(q-1)$	$-q+1$	$-p+1$	1	0

Here, $K_1 = \{id_{G_{i+5}}\}$, $K_2 = \cup_{i=1}^{l_1}(a^{v_i})^G$, $K_3 = \cup_{i=1}^{l_2}(b^{u_i})^G$, $K_4 = \cup_{i=1}^{l_3}(b^{u'_i}a^{v'_i})^G$ and $K_5 = \cup_{k=1}^{r-1}(c^k)^G$ are the rational conjugacy classes of G and $\gamma_1 = 1$, $\gamma_2 = \sum_{n=1}^{r-1}\chi_n$, $\gamma_3 = \sum_{s=1}^{l_3}\eta_s$, $\gamma_4 = \sum_{l=1}^{l_1}\theta_l$ and $\gamma_5 = \sum_{m=1}^{l_2}\varphi_m$.

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References

1. H. Behravesh, Quasi-permutation representations of $F_{p,q}$, *Vietnam J. Math.* **31** (2003), 249–254.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of Finite Groups, Oxford Univ. Press, Clarendon, Oxford, 1985.
3. H. Hölder, Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4, *Math. Ann.* **XLIII** (1893), 371–410.
4. I. M. Iscaacs, Character Theory of Finite Groups, Dover, New-York, 1976.
5. G. James, M. Liebeck, Representations and Characters of Groups, Cambridge Univ. Press, London-New York, 1993.
6. The Gap Team, GAP – Groups, Algorithms and Programming, Version 4.
7. T. Yamada, On the group algebras of metabelian groups over algebraic number fields I, *Osaka J. Math.* **6** (1969), 211–228.

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RATIONAL CHARACTER TABLE OF SOME FINITE GROUPS

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جدول سرشت‌های گویای برخی از گروه‌های متناهی

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در این مقاله، به محاسبه جدول سرشت‌های گروه دودوری T_{4n}، گروه‌های مرتبه pq و گروه‌های مرتبه pqr پرداخته‌ایم. هم‌چنین برخی از خواص جدول گویا-سرشت نیز آورده شده‌است.

کلمات کلیدی: جدول سرشت‌های گویا، جدول سرشت، گروه گالوا.