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MAGMA-JOINED-MAGMAS: A CLASS OF NEW ALGEBRAIC STRUCTURES

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Abstract. By left magma-e-magma, I mean a set containing a fixed element e, and equipped with the two binary operations "$\cdot$" and $\odot$, with the property of $e\odot(x\cdot y) = e\odot(x\odot y)$, namely the left e-join law. Thus $(X, \cdot, e, \odot)$ is a left magma-e-magma if and only if $(X, \cdot)$ and $(X, \odot)$ are magmas (groupoids), $e \in X$ and the left e-join law holds. The right and two-sided magma-e-magmas are defined in an analogous way. Also X is a magma-joined-magma if it is magma-x-magma for all $x \in X$. Therefore, I introduce a big class of basic algebraic structures with two binary operations, some of whose sub-classes are group-e-semigroups, loop-e-semigroups, semigroup-e-quasigroups and etc. A nice infinite (resp. finite) example of them is the real group-grouplike $(\mathbb{R}, +, 0, +_1)$ (resp. Klein group-grouplike). In this paper, I introduce and study the topic, construct several big classes of such algebraic structures and characterize all the identical magma-e-magmas in several ways. The motivation of this study lies in some interesting connections to f-multiplications, some basic functional equations on algebraic structures and Grouplikes (recently introduced by me). Finally, I present some directions for the researches conducted on the subject.

1. Backgrounds

A binary operation "$\cdot$" on a set X is a function that maps elements of the Cartesian product $X \times X$ to X. One of the most basic algebraic structures is a set equipped with an arbitrary binary operation, namely

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magma (groupoid or binary system; see [1, 2]). Thus magma is a basic type of algebraic structures with one binary operation. Here, I introduce a class of basic algebraic structures with two binary operations.
Motivations. There are three principle motivations for this topic, as follow:

(a) f-Multiplication of binary operations. Let $(X,\cdot)$ be a magma, and $f : X \to X$ be an arbitrary function. Another binary operation $\cdot_f$ in X is defined by $x \cdot_f y = f(xy)$, and is called f-multiplication of "$\cdot$". Thus I obtain the algebraic system $(X,\cdot,\cdot_f)$ whose binary operations "$\cdot$" and $\cdot_f$ may have some relations depending on the properties of "$\cdot$" and f (see [6]). For example, $(X,\cdot_f)$ is a semigroup if and only if f is associative in $(X,\cdot)$ (i.e., f satisfies the associative equation (2.9)).

(b) Some basic functional equations on algebraic structures. General solution of a functional equation on an algebraic structure is a challenging problem in the topic (see [3]). Let $(X,\cdot)$ be a magma, and consider the algebraic structures $(X,\cdot_f)$ and $(X,\cdot,\cdot_f)$. Many basic and important functional equations in $(X,\cdot)$ can be interpreted as a type or property of the two new algebraic systems, and vice versa, for instance:
- $(X,\cdot_f)$ is a semigroup if and only if f satisfies the associative equation $f(xf(yz)) = f(f(xy)z)$ (for all $x, y, z \in X$).
- $(X,\cdot_f)$ is commutative if and only if f satisfies the equation $f(xy) = f(yx)$.
- As we will see in this paper, $(X,\cdot,e,\cdot_f)$ is a left e-magmag (i.e., $e \cdot_f (x \cdot y) = e \cdot_f (x \cdot_f y)$ for all $x, y \in X$) if and only if f satisfies the following equation in $(X,\cdot)$

$$f(e(xy)) = f(ef(xy)) \;\; ; \;\; \forall x, y \in X. \tag{1.1}$$

Also, $(X,\cdot,e,\cdot_f)$ is a left identical e-magmag if and only if

$$f(e(xy)) = f(ef(xy)) = f(xy) \;\; ; \;\; \forall x, y \in X. \tag{1.2}$$

(c) Grouplikes and f-grouplikes. We observe that the second magma of every group-e-semigroup $(G,\cdot,e,\odot)$, where e is the identity of $(X,\odot)$, is a Grouplike that has been introduced in 2013 [4]. A grouplike is something between semigroup and group, and its axioms are generalizations of the four group axioms. Also every grouplike is a semigroup containing the minimum ideal that is also a maximal subgroup.

2. Magma-e-Magma and its related algebraic structures

Now let's introduce a class of new algebraic structures with two binary operations that is connected to magmas, semigroups, monoids, quasigroups, loops, groups, and other similar structures.

Definition 2.1. A set X with two binary operations "$\cdot$" and $\odot$ is called a *left magma-e-magma* (left e-magmag for short) if there is an element $e \in X$ satisfying the following property (called left e-join law)

$$e \odot xy = e \odot (x \odot y) \ \ ; \ \ \forall x, y \in X, \tag{2.1}$$

where $x \cdot y$ is denoted by xy, and $e \odot xy = e \odot (xy)$.

Thus $(X, \cdot, e, \odot)$ is a *left magma-e-magma* if and only if $(X, \cdot)$ and $(X, \odot)$ are magmas, $e \in X$, and the left e-join law holds. Therefore, this topic uses all algebraic structures with one binary operation, and introduces so many algebraic structures with two binary operations, the most basic structure of which is magma-e-magma.

If the first (second) magma of $(X, \cdot, e, \odot)$ belongs to some special class of magmas, we replace the first (the second) magma with the name of the class. For instance, a left monoid-e-epigroup is a left magma-e-magma $(X, \cdot, e, \odot)$, where $(X, \cdot)$ is a monoid, and $(X, \odot)$ is an epigroup. Right magma-e-magma is defined in an analogous way. A magma-e-magma is both a left and right magma-e-magma, and in such a case, e is called a joiner.

Now let X be a left magma-e-semigroup. Then

$$e \odot xy = e \odot x \odot y \ \ ; \ \ \forall x, y \in X, \tag{2.2}$$

and, using the associativity of $\odot$, we have

$$e\odot((xy)z) = e\odot((xy)\odot z) = (e\odot(xy))\odot z = (e\odot x\odot y)\odot z = e\odot x\odot y\odot z,$$

$$e \odot (x(yz)) = e \odot (x \odot (yz)) = e \odot x \odot (yz) \ \ ; \ \ \forall x, y, z \in X.$$

Hence, if it is semigroup-e-semigroup, then

$$e \odot (xyz) = e \odot x \odot y \odot z = e \odot (xy) \odot z = e \odot x \odot (yz) \ \ ; \ \ \forall x, y, z \in X,$$

and so

$$e \odot x_1 \cdots x_n = e \odot x_1 \odot \cdots \odot x_n \ \ ; \ \ \forall n \in \mathbb{N} \ , \ \forall x_1, \cdots , x_n \in X.$$

Therefore, if $x_1 = x_2 = \cdots = x_n = x$, then putting $x^n = xx \cdots x$ and $x^{\odot n} = x \odot x \odot \cdots \odot x$ (n times), we have $e \odot x^n = e \odot x^{\odot n}$ for every positive integer n and all $x \in X$.

Example 2.2. Consider an arbitrary non-empty set X.

(a) For every magma $(X,\cdot)$ and each $e \in X$, $(X,\cdot,e,\cdot)$ is an e-magmag (here $\odot = \cdot$), which is called *trivial e-magmag*. Thus in a trivial e-magmag, the types of the first and the second magmas are the same, e.g. group-e-group.

(b) Let $(X,\odot)$ be a magma with a left (resp. right, two-sided) zero 0. Then for every binary operation "$\cdot$" in X, $(X,\cdot,0,\odot)$ is a left (resp. right, two-sided) 0-magmag.

(c) Define $x \odot y = x$ (resp. $x \odot y = y$) for every $x, y \in X$. Then for every $e \in X$ and all binary operations, "$\cdot$" in X $(X,\cdot,e,\odot)$ is a left (resp. right) magma-e-semigroup. It is not (two-sided) magma-e-semigroup if $\cdot \neq \odot$.

(d) Consider the additive group $(\mathbb{R},+)$, and fix $b \in \mathbb{R}\setminus\{0\}$. For each real number a, denote by $[a]$ the largest integer not exceeding a, and put $(a) = a - [a] = \{a\}$ (namely decimal or fractional part of a). Set

$$[a]_b = b[\frac{a}{b}] \ \ , \ \ (a)_b = b(\frac{a}{b}).$$

We call $[a]_b$ the *b-integer part of a*, and $(a)_b$ *the b-decimal part of a*. Also $[\ ]_b$ and $(\ )_b$ are called the *b-decimal part function and b-integer part function*, respectively (see [7]). Since $(a)_1 = (a)$, we may use the symbol $(a)_1$ for the decimal part of a. Now for every $x, y \in \mathbb{R}$, we put $x +_b y = (x+y)_b$, and call $+_b$ *b-addition*. The binary system $(\mathbb{R},+_b)$ is a special case of semigroups, namely real b-grouplike (see [4]). We claim that the structure $(\mathbb{R},+,e,+_b)$ is a group-e-semigroup for all $e \in \mathbb{R}$ because

$$\begin{aligned} e +_b (x +_b y) &= (e + (x+y)_b)_b = (e + x + y)_b \\ &= e +_b (x+y) \ \ ; \ \ \forall x, y \in \mathbb{R}. \end{aligned}$$

If $e = 0$, then we call it *real b-group-grouplike*, and specially, *real group-grouplike* if $b = 1$.

(e) For an example of finite (nontrivial) e-magmags, consider $K = \{e, a, \eta, \alpha\}$, and define binary operations "$\cdot$" and $\odot$ by the following multiplication tables

$\cdot$	e	a	η	α
e	e	a	η	α
a	a	e	α	η
η	η	α	e	a
α	α	η	a	e

$\odot$	e	a	η	α
e	e	a	e	a
a	a	e	a	e
η	e	a	e	a
α	a	e	a	e

Note that $(K,\cdot)$ is the Klein four-group and $(K,\odot)$ is the Grouplike introduced in [4, example 2.4]. It is easy to see that k-join law holds, for every $k \in K$. Thus we call $(K, \cdot, e, \odot)$ *Klein group-grouplike*.

In this section, I often focus on the left structures (because of similarities). In this way, I first consider the basic properties of left e-magmags and try to put everything in its most natural framework, and then study more useful structures such as group-e-semigroups, characterize their identical classes, and determine their general form.

Let X be a magma, and $e \in X$. We call X, *left e-unital* (resp. *right e-unital*) if e is a left (resp. right) identity of X. Hence, X is *e-unital* if it is left and right e-unital. If X is e-unital, then e is the unique (two-sided) identity of X, and sometimes we use the notation $(X, \cdot, e)$ for it (so it is usable for monoid, loop, and group). Now, if $Y \subseteq X$ and $ey = y$ for every $y \in Y$, then we can say that e is a left identity of Y in X. If this is the case and $e \in Y$, then we call e a left identity of Y. Also, e is called *left bi-identity* (resp. *middle bi-identity* or *middle identity*) of X if $e(xy) = (ex)y = xy$ (resp. $(xe)y = x(ey) = xy$) for every $x, y \in X$. Analogously, the right and two-sided cases are defined.

Note. It is worth noting that for every non-trivial left magma-e-magma $(X, \cdot, e, \odot)$, the second magma $(X, \odot)$ can not be left e-unital. Thus, in its title, the second magma can not be replaced by monoid, loop or group. In fact, if e is a left identity of $XX \cup X \odot X$ in $(X, \odot)$, then $xy = e \odot xy = e \odot (x \odot y) = x \odot y$ (for every $x, y \in X$), and so $\cdot = \odot$.

Now I introduce a required definition for magmas, which is used in the topic repeatedly.

Definition 2.3. Let e be a fixed element of the magma X. We call X *e-associative* if the equation $x(yz) = (xy)z$ holds for every x, y, z of X such that at least one of them is equal to e. Also we call X *weakly e-associative* if the equation holds whenever $x = z = e$.

Hence, every e-unital magma is e-associative, and X is a semigroup if and only if it is e-associative for every $e \in X$.

Now, consider the left e-magmag $(X, \cdot, e, \odot)$. If $(X, \cdot)$ is left (resp. right) e-unital, then $e \odot x = e \odot (e \odot x)$ (resp. $e \odot x = e \odot (x \odot e)$) for every $x \in X$. Thus if $(X, \cdot)$ is left e-unital, then e is a left identity of $e \odot X$ in $(X, \odot)$, and if $(X, \cdot)$ is right e-unital and $(X, \odot)$ is weakly e-associative, i.e.

$$e \odot (x \odot e) = (e \odot x) \odot e \ ; \ \forall x \in X, \tag{2.3}$$

then e is a right identity of $e \odot X$ in $(X, \odot)$.

Therefore, if the first magma is e-unital and the second one is weakly e-associative, then e is an identity of $e\odot X$ in $(X,\odot)$, and so

$$e\odot(e\odot x)=e\odot x=(e\odot x)\odot e \ \ ; \ \ \forall x\in, \qquad (2.4)$$

Specially, $e^{\odot 3}$ is well-defined (i.e. $e\odot(e\odot e)=(e\odot e)\odot e$) and $e^{\odot 3}=e^{\odot 2}$. This discussion leads us to the following definition and lemma.

Definition 2.4. Let $(X,\cdot,e,\odot)$ be a left magma-e-magma. We call it

(a) *unital left magma-e-magma* or briefly *left e-unimag*, if the first magma $(X,\cdot)$ is e-unital.
(b) *left magma-e-ass.magma* (resp. *left magma-e-weakass.magma*) or briefly *left e-assmag* (resp. *left e-wassmag*), if the second magma $(X,\odot)$ is e-associative (resp. weakly e-associative).

Also every *unital left magma-e-ass.magma* (resp. *unital left magma-e-weakass.magma*) is called *left e-uniassmag* (resp. *left e-uniwassmag*), briefly. (The right and two-sided cases are defined, similarly.)

Thus we have several special (left) e-magmags such as (left) e-unimag, e-assmag, e-uniassmag, and e-uniwassmag that are more general than (left) monoid-e-semigroup, loop-e-semigroup, and group-e-semigroup.

Note. As one can see in the above definition and previous explanation, some conditions or properties of an e-magmag may be about the first or the second magma, only. Hence, if $\mathcal{C}$ is a condition (or property), by *first-$\mathcal{C}$ e-magmag* (resp. *second-$\mathcal{C}$ e-magmag*), we mean an e-magmag such that its first (resp. second) magma has the condition $\mathcal{C}$. For example, second-associative left e-magmag is the same left magma-e-semigroup, and every e-uniassmag is first-e-unital and second-e-associative. This general naming method helps us to call and consider so many types of e-magmags.

Now considering the previous discussion and definitions:

- If $(X,\cdot,e,\odot)$ is left e-uniwassmag, then e is an identity of $e\odot X$ in $(X,\odot)$.
- If $(X,\cdot,e,\odot)$ is left e-assmag, then its left e-join law can be written as equation (2.2). Moreover, if $e\odot(xe)=e\odot x$ (e.g. if $(X,\cdot)$ is right e-unital), then

$$(e\odot x)\odot(e\odot y)=((e\odot x)\odot e)\odot y=(e\odot(x\odot e))\odot y$$

$$=(e\odot(xe))\odot y=(e\odot x)\odot y=e\odot(x\odot y)=e\odot xy,$$

thus

$$e\odot xy=e\odot x\odot y=(e\odot x)\odot(e\odot y) \ \ ; \ \ \forall x,y\in X. \qquad (2.5)$$

The above identity has more properties than equation (2.2), and it implies that $e \odot X$ is a right $e \odot e$-unital submagma of $(X, \odot)$, and so $e \odot e$ is its idempotent.

Therefore, we arrive at the following important properties for left e-uniassmags and magma-e-semigroup. Note that, although left e-uniassmags (a left e-magmag such that its first magma is e-unital and the second one is e-associative) have a weaker condition than unital left magma-e-semigroups, they have many important properties, and there are useful basic structures in the topic.

Lemma 2.5. *(i) If $(X, \cdot, e, \odot)$ is a left e-assmag such that $e \odot (xe) = e \odot (ex) = e \odot x$ (e.g. if $(X, \cdot, e, \odot)$ is left e-uniassmag), then $(e \odot X, \odot)$ is an $e \odot e$-unital sub-magma.*
(ii) If $(X, \cdot, e, \odot)$ is a unital left magma-e-semigroup, then $(e \odot X, \odot)$ is a monoid. Moreover, if the equation $xy = e$ $(yx = e)$ has a solution in X for every fixed element $x \in X$, then $(e \odot X, \odot)$ is a group.

Proof. (i) Considering equation (2.5), it is enough to show that $e \odot X$ is $e \odot e$ -unital.

Putting $x = e$ (resp. $y = e$) in the identities, we conclude that $e \odot e$ is a left (resp. right) identity of the sub-magma $e \odot X$. Therefore, part (i) is proved.

(ii) The first part implies that $e \odot X$ is a sub-semigroup of $(X, \odot)$ with the identity element $e \odot e$. Now if $x \in X$ and y is the element, where $xy = e$, then

$$(e \odot x) \odot (e \odot y) = e \odot xy = e \odot e,$$

(analogously for another case). Hence, the proof is complete. □

Convention. In this paper, if the first magma of $(X, \cdot, e, \odot)$ is group, monoid or loop, then we suppose that e is the identity element of $(X, \cdot)$, and so the e-magmag is unital. Hence, for example, by left group-e-semigroup X we mean $(X, \cdot, e)$ is a group, $(X, \odot)$ is a semigroup, and the left e-join law holds.

2.1. **Essential map on e-magmags.** Let $(X, \cdot, e, \odot)$ be a left or right e-magmag. Then we have the maps $J_e = J_e^{\ell}, J^e = J_e^r$ from X into X defined by

$$J_e(x) = e \odot x \ , \ J^e(x) = x \odot e,$$

namely *left , right e-joiner map* (or e-map) , respectively. If $J_e = J^e$ (equivalently, $e \in Z(X, \odot)$), then we only use the notation J_e for them. Notice that the left (resp. right) e-map is the same left (resp. right) e-translation map in $(X, \odot)$. If X is a set with two binary operations "$\cdot$" and $\odot$, then $(X, \cdot, e, \odot)$ is left (resp. right) e-magmag if and only if

$J_e(xy) = J_e(x \odot y)$ (resp. $J^e(xy) = J^e(x \odot y)$) for all $x, y \in X$. Thus, if X is a left e-magmag, then the following statements are equivalent:
- J_e is left e-periodic with respect to "$\cdot$" (i.e. $J_e(ex) = J_e(x)$ for all x);
- J_e is left e-periodic with respect to $\odot$ (i.e. $J_e(e \odot x) = J_e(x)$ for all x);
- J_e is left e-periodic with respect to both binary operations (i.e. $J_e(ex) = J_e(x) = J_e(e \odot x)$ for all x);
- $e \odot (e \odot x) = e \odot x$ for all x;
- e is a left identity of $e \odot X$ in $(X, \odot)$;
- J_e is idempotent.

(Analogously, for the right and two-sided cases.) The e-maps play an important role for the study and characterization of such algebraic structures. For example, $(X, \odot)$ is weakly e-associative if and only if $J_e J^e = J^e J_e$ (i.e. the composition is commutative). Depending on each one of the first and the second binary operations, the e-map has important different properties, one of which is related to the decomposer, associative and canceler properties introduced and studied in [5]. Here, we give a short explanation about them.

If X is an arbitrary magma, then we have the following type functions:

$$f(f(x)f_*(y)) = f(x) \quad : \quad \text{left decomposer} \tag{2.6}$$

$$f(xf_*(y)) = f(x) \quad : \quad \text{left strong decomposer} \tag{2.7}$$

$$f(f(x)y) = f(xy) \quad : \quad \text{left canceler} \tag{2.8}$$

$$f(f(xy)z) = f(xf(yz)) \quad : \quad \text{associative} \tag{2.9}$$

$$f(f(xy)z) = f(xf(yz)) = f((xy)z) = f(x(yz)) \quad : \quad \text{strongly associative.} \tag{2.10}$$

Note that $f : X \to X$, $f(x)f_*(x) = x = f^*(x)f(x)$ for all $x \in X$, and the right cases are defined analogously. The function f is decomposer (resp. canceler) if it is both left and right decomposer (resp. canceler).

If $X = S$ is semigroup, then every canceler function is strongly associative, but the converse is true if f is periodic (i.e. there exists $T \in X$ such that $f(Tx) = f(xT) = f(x)$, for all $x \in X$) and idempotent. Of course, if $X = M$ is monoid, then f is strongly associative if and only if it is canceler.

If $X = G$ is group, then the following conditions are equivalent :

(i) f is strongly associative;
(ii) f is associative and idempotent;
(iii) f is associative and $f^2(e) = f(e)$;
(iv) f is associative and $f(e)$-periodic (i.e. $f(f(e)x) = f(xf(e)) = f(x)$);

(v) f is strong decomposer;
(vi) f is decomposer and $f^*(G) \trianglelefteq G$ or $f_*(G) \trianglelefteq G$;
(vii) f is decomposer and $f^*(G) = f_*(G) \trianglelefteq G$;
(viii) f is canceler.
(ix) There exist $\Delta \trianglelefteq G$ and $\Omega \subseteq G$ such that $G = \Delta \cdot \Omega = \Delta \cdot \Omega$ and $f = P_\Omega$ (where the product is direct, and P_Ω is the projection map from G onto Ω).

The last item characterizes strong decomposer, canceler, and strongly associative functions on groups ([5]).

Example 2.6. The b-decimal part function $(\ )_b$ satisfies the above equivalent conditions in the additive real numbers group. Here, $\Delta = b\mathbb{Z}$, $\Omega = b[0,1) = \mathbb{R}_b$, and we have $\mathbb{R} = b\mathbb{Z}\dot{+}b[0,1)$, $(\ )_b = P_{b[0,1)}$ ($\dot{+}$ means the sum is direct; see [5, 7]).

Now let's show that e-map has the mentioned and some other properties in e-assmags and e-uniassmags.

Lemma 2.7. *Assume $(X, \cdot, e, \odot)$ is a left e-assmag.*

(A) *If J_e is right e-periodic, then:*
$J_e : (X, \cdot) \to (X, \odot)$ is homomorphism,
$J_e : (X, \odot) \to (X, \odot)$ is endomorphism and right canceler,
$J_e : (X, \cdot) \to (X, \cdot)$ is right canceler.

(B) *If J_e is left e-periodic, then J_e is left canceler from the both magmas to themselves.*
(C) *Hence, if X is a left e-uniassmag, then J_e is e-periodic, homomorphism from both magmas into the second magma and canceler as a function from both magmas to themselves, and so:*

$$\begin{aligned} J_e(xy) = J_e(x \odot y) = J_e(x) \odot J_e(y) = J_e(J_e(x)y) = J_e(xJ_e(y)) \\ = J_e(J_e(x) \odot y) = J_e(x \odot J_e(y)) \ \ ; \ \ \forall x, y \in X. \end{aligned} \tag{2.11}$$

(D) *If X is a unital left magma-e-semigroup, then all of the above properties hold, and $J_e : (X, \cdot) \to (X, \cdot)$ is strongly associative.*

Proof. Let $(X, \cdot, e, \odot)$ be a left e-assmag. First note that the condition $e \odot (xe) = e \odot x$ (resp. $e \odot (ex) = e \odot x$) for equation (2.5), equivalent to J_e, is right (resp. left) e-periodic. Thus if J_e is right e-periodic, then equation (2.5) holds, and for every x, y:

$$J_e(xJ_e(y)) = e \odot (x(e \odot y)) = e \odot (x \odot (e \odot y))$$

$$= (e \odot x) \odot (e \odot y) = e \odot x \odot y = J_e(xy).$$

Also if J_e is left e-periodic (or equivalently idempotent), then:

$$J_e(J_e(x)y) = e \odot ((e \odot x)y) = e \odot ((e \odot x) \odot y)$$
$$= (e \odot (e \odot x)) \odot y = e \odot x \odot y = J_e(xy).$$

Note that the identity (2.5) is equivalent to

$$J_e(xy) = J_e(x \odot y) = J_e(x) \odot J_e(y) \ \ ; \ \ \forall x, y \in X.$$

and if X is e-uniassmag, then J_e is e-periodic, obliviously.

Applying the above facts, we arrive at (A), (B), and (C). Finally, if X is unital left magma-e-semigroup, then J_e is canceler, and so:

$$J_e(J_e(xy)z) = J_e((xy)z) = J_e(x(yz)) = J_e(xJ_e(yz)).$$

Hence, J_e is strongly associative.

□

Example 2.8. Consider the real b-group-grouplike $(\mathbb{R}, +, 0, +_b)$. Then $J_0 = (\ )_b$ agrees with Example 2.6 and Lemma 2.7.

Define $f : K \to K$ by $f(\eta) = f(e) = e$, $f(\alpha) = f(a) = a$ (K is the Klein group-grouplike). Then, clearly, we have $f_*(a) = f_*(e) = e$, $f_*(\alpha) = f_*(\eta) = \eta$, and $f_* = f^*$. One can easily check that f is strongly associative, strong decomposer, e, η-periodic, and idempotent from both magmas to themselves. As we mentioned earlier, since $(K, \cdot)$ is group, then $f^*(K) = f_*(K) = \{e, \eta\} \trianglelefteq K$, $f(K) = \{e, a\}$ and $(K, \cdot) = \{e, \eta\} \cdot \{e, a\}$, where the product is direct. Now we have $f = P_{\{e,a\}} = J_e$, which agrees with the theorem and all the mentioned properties.

2.2. **Sub-e-magmags and e-magmag homomorphisms.** A (left) sub-magma-e-magma (sub-e-magmag) of $(X, \cdot, e, \odot)$ is a subset Y such that it itself is a left e-magmag with the restrictions of the binary operations, and which contains the joiner e. Thus Y is a sub-e-magmag of X if and only if Y is submagma of X with both binary operations and $e \in Y$.

Now, let $f : (X, \cdot) \to (X', \cdot')$, $f : (X, \odot) \to (X', \odot')$ be magma homomorphisms. If $(X, \cdot, e, \odot)$ is left e-magmag, then $(f(X), \cdot', f(e), \odot')$ is a left $f(e)$-magmag. Thus if f is onto, then putting $e' = f(e)$, $(X', \cdot', e', \odot')$ is also left e'-magmag. Therefore, we call $f : (X, \cdot, e, \odot) \to (X', \cdot', e', \odot')$ an e-magmag homomorphism if and only if :

(i) $f : (X, \cdot) \to (X', \cdot')$ is homomorphism,
(ii) $f : (X, \odot) \to (X', \odot')$ is homomorphism,
(iii) $f(e) = e'$.

We say that $(X, \cdot, e, \odot)$ is isomorphic to $(X', \cdot', e', \odot')$, and denote by

$$(X, \cdot, e, \odot) \cong (X', \cdot', e', \odot'),$$

if there exists an isomorphism (bijective homomorphism) between them. If $(X,\cdot,e,\odot)\cong(X',\cdot',e',\odot')$, then $(X,\cdot)\cong(X',\cdot')$ and $(X,\odot)\cong(X',\odot')$ (but the converse is not true).

For every (left, right and two-sided) e-magmag homomorphism $f:(X,\cdot,e,\odot)\to(X',\cdot',e',\odot')$, we have two useful kernels, as follows:

$$\ker(f)=\ker_e(f)=\{x\in X: f(x)=f(e)\};$$
$$\ker_{e\odot e}(f)=\{x\in X: f(x)=f(e\odot e)\}.$$

(Note that $f(e)=e'$ and $f(e\odot e)=e'\odot' e'$.) We call $\ker_e(f)$ (resp. $\ker_{e\odot e}(f)$) e-kernel (resp. $e\odot e$-kernel) of f. It is easy to see that $Im(f)$ is a sub-e'-magmag of X', and $\ker_e(f)$ is a sub-e-magmag of X if e' is idempotent with respect to both binary operations.

Remark 2.9. Assume $(X,\cdot,e,\odot)$ is a left e-assmag, for which J_e is right e-periodic. Lemma 2.7 implies that J_e is a left e-magmag homomorphism from $(X,\cdot,e,\odot)$ to the trivial left $e\odot e$-magmag $(X,\odot,e\odot e,\odot)$. Since, $e\odot e$ is $\odot$-idempotent (i.e. idempotent with respect to $\odot$), then:

$$\ker(J_e)=\{x\in X: J_e(x)=e\odot e\}=\{x\in X: e\odot x=e\odot e\}$$

is a left sub-$e\odot e$-magmag of $(X,\odot,e\odot e,\odot)$. Hence, if X is left e-uniassmag, then $(X,\cdot)$ and $(e\odot X,\odot)$ are unital magmas, and $J_e:(X,\cdot)\to(e\odot X,\odot)$ is a unital magma epimorphism (which maps the first identity to the second one).

Example 2.10. Considering the above remark, $(\mathbb{R}_b,+_b,0,+_b)$ is a subgroup-0-group of $(\mathbb{R},+_b,0,+_b)$, and $(\ )_b:(\mathbb{R},+)\to(\mathbb{R}_b,+_b)$ is a group epimorphism with kernel $\{x\in\mathbb{R}: 0+_b x=(x)_b=0+_b 0=0\}=b\mathbb{Z}$.

2.3. e-Relation, e-congruence and its induced quotient magma.

Here, we consider two essential equivalence relations on every left and right e-magmags (namely *left and right e-relation*) that are completely connected to the left and right essential e-maps.

For every left or right e-magmag X, we define $x\sim_e y$ (resp. $x\sim^e y$) if and only if $e\odot x=e\odot y$ (resp. $x\odot e=y\odot e$). It is clear that $x\sim_e y$ and $x\sim^e y$ are two equivalence relations in X and $x\sim_e y$ (resp. $x\sim^e y$) if and only if $J_e(x)=J_e(y)$ (resp. $J^e(x)=J^e(y)$).

Now let X be a left e-magmag. The equivalence relation $\sim_e$ is a magma congruence with respect to both binary operations if X is left e-assmag and J_e is right e-periodic. Since $x_1\sim_e y_1$ and $x_2\sim_e y_2$, then:

$$x_1x_2\sim_e x_1\odot x_2\sim_e y_1\odot y_2\sim_e y_1y_2;$$

in fact, by applying equation (2.5)

$$e\odot x_1x_2=e\odot(x_1\odot x_2)=(e\odot x_1)\odot(e\odot x_2)=(e\odot y_1)\odot(e\odot y_2)$$

$$= e \odot (y_1 \odot y_2) = e \odot y_1 y_2.$$

Thus if X is left e-assmag and J_e is right e-periodic, then we have two binary operations in $\overline{X} = X/\sim_e$ (the set of all congruence classes $\overline{x}$), defined by $\overline{x}\,\overline{y} := \overline{xy}$ and $\overline{x} \odot \overline{y} := \overline{x \odot y}$. But the two binary operations are the same because $x \odot y \sim_e xy$, and

$$\overline{x} \odot \overline{y} = \overline{x \odot y} = \overline{xy} = \overline{x}\,\overline{y}.$$

Therefore, the induced left quotient $\overline{e}$-magmag is trivial, and we have only $\overline{e}$-associative quotient magma $\overline{X}$ with a right $\overline{e}$-periodic map $J_{\overline{e}} = \overline{J_e}$:

$$J_{\overline{e}}(\overline{x}) = \overline{e}\,\overline{x} = \overline{ex} = \overline{e \odot x} = \overline{J_e(x)}.$$

Hence, if X is left e-assmag and J_e is e-periodic (e.g. if X is left e-uniassmag), then $\overline{X} = X/\sim_e$ is an $\overline{e}$-unital and $\overline{e}$-associative magma.

Identity and Associativity of a magma up to e-relation. First note that the e-relations $\sim_e$ and $\sim^e$ can be defined in every arbitrary magma containing e. Now let $(X, \cdot)$ be a magma containing the fixed element e. We say that "$(X, \cdot)$ is left e-unital up to the left e-relation" or simply "$(X, \cdot)$ is left $\sim_e$-unital" if $ex \sim_e x$ for every $x \in X$ (analogously, for the right and two-sided cases). Also we say that "$(X, \cdot)$ is associative up to the left e-relation" or simply "$(X, \cdot)$ is $\sim_e$- associative" if $x(yz) \sim_e (xy)z$ for every $x, y, z \in X$.

It is interesting to know that if X is left e-magmag, then:

$$(X, \cdot) \text{ is left} \sim_e\text{-unital} \Leftrightarrow J_e \text{ is left } e\text{-periodic} \Leftrightarrow (X, \odot) \text{ is left} \sim_e\text{-unital}$$

$$\Leftrightarrow J_e \text{ is idempotent} \Leftrightarrow e \text{ is a left identity of } e \odot X \text{ in } (X, \odot).$$

Also we have:

$$(X, \cdot) \text{ is right } \sim_e\text{-unital} \Leftrightarrow J_e \text{ is right } e\text{-periodic} \Leftrightarrow J_e J^e = J_e$$

$$\Leftrightarrow (X, \odot) \text{ is right } \sim_e\text{-unital}.$$

Remark 2.11. In this section, one can see the condition "J_e is right e-periodic", repeatedly. Hence, the condition that can be replaced by "$(X, \cdot)$ is right $\sim_e$-unital" (or other equivalent conditions) that is weaker than the condition "$(X, \cdot)$ is right e-unital". Therefore, if X is a left e-assmag such that $(X, \cdot)$ is right $\sim_e$-unital, then the quotient magma $\overline{X} = X/\sim_e$ exists; it is $\overline{e}$-associative and right $\overline{e}$-unital, and also $(e \odot X, \odot)$ is a right $e \odot e$-unital magma, and we have the maps in the next part. Hence, if X is a left e-assmag such that $(X, \cdot)$ is $\sim_e$-unital (that is weaker than the condition "X is left e-uniassmag"), then we have all the above items, and also two unital magmas that are considered in the next important theorem.

Other important maps on e-magmags. Up to now, we have introduced essential e-maps on (left and right) e-magmags. But we can define some other useful maps, as follow:
As we know, for any arbitrary left e-magmag X, we have the map $J_e : X \to X$ with $Im(J_e) = e \odot X$, and the equivalence relation $\sim_e$ with the partition $\overline{X} = X/\sim_e$, and

$$\overline{x} = x/\sim_e = J_e^{-1}(\{J_e(x)\}) \quad : \quad \forall x \in X.$$

Now we can define the other useful well-defined maps, and consider the relations between the image and partition:
- As usual, the (natural) surjective map $\pi_e : X \to \overline{X}$ is defined by $\pi_e(x) = \overline{x}$ with the kernel (only as a map):

$$\ker(\pi_e) := \{x \in X : \pi_e(x) = \pi_e(e)\} = \overline{e} \tag{2.12}$$
$$= \{x \in X : e \odot x = e \odot e\} = \ker_{e\odot e}(J_e).$$

- The injective map $\lambda_e : \overline{X} \to X$ is defined by $\lambda_e(\overline{x}) := e \odot x = J_e(x)$.
- Since $\lambda_e : \overline{X} \to e \odot X$ is bijection and invertible, then we have the bijective map $\phi_e := \lambda_e^{-1} : e \odot X \to \overline{X}$ with $\phi_e(e \odot x) = \overline{x} = \pi_e(x)$.

Therefore, we have the following chain maps, composition relations images, and kernel:

$$X \xrightarrow{J_e} e \odot X \xrightarrow{\phi_e} \overline{X} \xrightarrow{\lambda_e} X \xrightarrow{\pi_e} \overline{X}, \tag{2.13}$$
$$e \odot X \xrightarrow{\phi_e} \overline{X} \xrightarrow{\lambda_e} X \xrightarrow{J_e} X \xrightarrow{\pi_e} \overline{X},$$

$$\pi_e\lambda_e = \phi_e\lambda_e = \iota_{\overline{X}} \ , \ \lambda_e\phi_e = \iota_{e\odot X} \ , \ \lambda_e\pi_e = J_e \ , \ \phi_e J_e = \pi_e, \tag{2.14}$$

$$\begin{gathered} Im(\phi_e) = Im(\pi_e) = \overline{X} \ , \ Im(\lambda_e) = Im(J_e) = e \odot X, \\ \ker(\lambda_e) := \{\overline{x} \in \overline{X} : \lambda_e(\overline{x}) = \lambda_e(\overline{e})\} = \{\overline{e}\} \end{gathered} \tag{2.15}$$

Thus:

$$Card(\overline{X}) = Card(e \odot X) = Card(J_e(X)).$$

Now if X is left $\sim_e$-unital, then $\overline{e \odot x} = \overline{x}$ (for all x), and so are $\pi_e J_e = \pi_e$ and $\phi_e = \pi_e|_{e\odot X}$.

But, if X is e-assmag and the magma is right $\sim_e$-unital, then all the mentioned maps are also homomorphism, J_e is canceler, and we have the right $\overline{e}$-unital quotient magma $\overline{X} = X/\sim_e$ and the right $e\odot e$-unital magma $e \odot X$. Therefore, if X is (two-sided) $\sim_e$-unital e-assmag (e.g. if X is e-uniassmag), then all of the above properties hold. Therefore, we arrive at the following important theorem.

Theorem 2.12. *If X is a left e-assmag, for which one of its magmas is right $\sim_e$-unital (resp. left e-uniassmag), then the two right unital magmas (resp. unital magmas) $X/\sim_e$ and $e \odot X$ are isomorphic.*

Hence, if X is unital left magma-e-semigroup (resp. loop-e-semigroup), then $X/\sim_e \cong e\odot X$ as two monoids (resp. groups).

Proof. Applying the above discussions, maps, identities, lemma 2.7, and Remark 2.9, one can prove this theorem. ☐

Example 2.13. Consider the real b-group-grouplike $(\mathbb{R},+,0,+_b)$. We have

$$x\sim_b y \Leftrightarrow 0+_b x = 0+_b y \Leftrightarrow (x)_b=(y)_b \Leftrightarrow x-y\in b\mathbb{Z} \Leftrightarrow x\equiv y \pmod b$$

Also

$$\overline{x}+_b\overline{y}=\overline{x+_b y}=\overline{(x+y)_b}=\overline{x+y}=\overline{x}+\overline{y},$$

which agrees with the previous results, and so:

$$(\mathbb{R},+_b)/\sim_b=(\mathbb{R},+)/\sim_b\cong \mathbb{R}/b\mathbb{Z},$$

where $\mathbb{R}/b\mathbb{Z}$ is the quotient group $\mathbb{R}$ over the cyclic subgroup $b\mathbb{Z}=\langle b\rangle$. On the other hand,

$$0+_b\mathbb{R}=\{(x)_b|x\in\mathbb{R}\}=b[0,1)=\mathbb{R}_b,$$

and the above theorem implies that $\mathbb{R}/b\mathbb{Z}\cong(\mathbb{R}_b,+_b)$, which is the reference b-bounded group (the group of all least nonnegative (real) residues mod b, if $b>0$) introduced, and studied in [7]. Here, $J_e=J_0=(\ )_b=P_{b[0,1)}$ (see Example 2.6) and

$$\pi_0(x)=\overline{x}=x+b\mathbb{Z}\ ,\ \lambda_0(\overline{x})=\lambda_0(x+b\mathbb{Z})=0+_b x=(x)_b=J_0(x)$$

$$\phi_0(0+_b x)=\phi_0((x)_b)=x+b\mathbb{Z}=\pi_0(x).$$

3. Identical e-Magmags, Magma-joined-Magmas, and Their Characterization

Now, we introduce some types of e-magmags X, where the e-join law holds for every $e\in X$. At first, recall that a magma $(X,\cdot)$ is called surjective if the function $\cdot: X\times X\to X$ is surjective (i.e. every $x\in X$ can be written as $x=yz$ for some $y,z\in X$). Hence, X is surjective if and only if $XX=X$. It is obvious that if $(X,\cdot)$ is left or right unital, then X is surjective. Thus all monids, loops, and groups are surjective.

Definition 3.1. We call the left e-magmag $(X,\cdot,e,\odot)$ *identical* if e is a left bi-identity of $(X,\odot)$.

Note that $(X,\cdot,e,\odot)$ is left identical e-magmag if and only if

$$e\odot xy=e\odot(x\odot y)=x\odot y\ \ ;\ \ \forall x,y\in X\ \ \text{(left identical } e\text{-join law)} \tag{3.1}$$

Therefore, a left e-magmag X is identical if and only if $J_e(xy)=x\odot y$ or equivalently $J_e(x\odot y)=x\odot y$ for all $x,y\in X$. If X is left identical

e-magmag, then both magmas are left $\sim_e$-unital, $J_e(xe) = J^e(x)$, and so all the equivalent conditions hold. Also, $(X, \odot)$ is semigroup if and only if $(X, \odot)$ is $\sim_e$-associative.

Example 3.2. The real b-group-grouplike is identical group-0-semigroup. In fact, it is identical group-β-semigroup, for every $\beta \in b\mathbb{Z}$. If $\beta = bk$, where $k \in \mathbb{Z}$, then:

$$\beta +_b (x + y) = \beta +_b x +_b y = (kb + x + y)_b = (x + y)_b = x +_b y.$$

The Klien group-grouplike is identical group-e-semigroup and group-η-semigroup.

Using the e-map and J_e-multiplication, we can give an interesting interpretation of identical e-magmags and basic idea for their characterization. First, we need to recall f-multiplication (introduced in [6]). The associative functions have so close relations to (associative) f-multiplication of binary operations that induce a semigroup.

Now we are ready to show the basic role of e-map and J_e-multiplication for identical e-magmags.

Let $(X, \cdot, e, \odot)$ be a left e-magmag. Then:

(a) $\cdot_{J_e} = \odot_{J_e}$ (conversely, if $\cdot_{J_e} = \odot_{J_e}$, then $(X, \cdot, e, \odot)$ is a left e-magmag).

(b) X is left identical e-magmag if and only if $\odot = \cdot_{J_e}$ and if and only if $\odot = \odot_{J_e}$ (equivalently $\odot = \cdot_{J_e} = \odot_{J_e}$); hence if $f = J_e$ and X is left identical e-magmag, then $\odot = \cdot_f$.

Conversely, if $\odot = \cdot_f$, then (for construction and characterization of the identical e-magmags and their second binary operation), we want to determine the condition for $f = J_e$.

For answering the question with the weakest conditions, we give a new discussion, as follow:

Let $(X, \cdot)$ be a magma containing the fixed element e and $f : X \to X$. Define f_e by $f_e(x) = f(ex)$, for all $x \in X$. Then for every $x, y \in X$, we have:

$$\begin{aligned} \cdot_{f_e} = \cdot_f &\Leftrightarrow x \cdot_{f_e} y = x \cdot_f y \;\Leftrightarrow f_e(xy) = f(xy) \\ \Leftrightarrow f(e(xy)) = f(xy) &\;\Leftrightarrow f|_{XX}(e(xy)) = f|_{XX}(xy) \\ &\Leftrightarrow\; f|_{XX} \text{ is left } e\text{-periodic.} \end{aligned} \tag{3.2}$$

Hence, if $XX = X$, then:

$$\cdot_{f_e} = \cdot_f \Leftrightarrow\; f \text{ is left } e\text{-periodic} \Leftrightarrow f_e = f. \tag{3.3}$$

Now let X be an arbitrary set with two binary operations $\cdot$, $\odot$ and $e \in X$. If $\odot = \cdot_f$, for some function $f : X \to X$, then $f_e = J_e$. If

$x \in X$, then:

$$J_e(x) = e \odot x = e \cdot_f x = f(ex) = f_e(x).$$

Therefore,

$$\odot = \cdot_f = \cdot_{f_e} \Leftrightarrow \odot = \cdot_f \ , \ f|_{XX} \text{ is left } e\text{-periodic} \qquad (3.4)$$
$$\Leftrightarrow \odot = \cdot_{f_e} \ , \ f|_{XX} \text{ is left } e\text{-periodic.}$$

Note that these equivalent conditions (3.4) imply that $f_e = J_e = (f_e)_e$ and $f|_{XX}, f_e$ are left e-periodic (because $\odot = \cdot_f$ (resp. $\odot = \cdot_{f_e}$), implying that $f_e = J_e$ (resp. $(f_e)_e = J_e$), and a function g is left e-periodic if and only if $g = g_e$). Hence, if $XX = X$, then:

$$\odot = \cdot_f = \cdot_{f_e} \Leftrightarrow \odot = \cdot_f \ , \ f = J_e \Leftrightarrow \odot = \cdot_f \ , \ f = f_e \qquad (3.5)$$
$$\Leftrightarrow \odot = \cdot_f \ , \ f \text{ is left } e\text{-periodic.}$$

Thus we obtain the following important lemma.

Lemma 3.3. *Let X be an arbitrary set containing e with two binary operations "$\cdot$" and $\odot$ such that "$\cdot$" is surjective, and let $f : X \to X$.*

(a) *If $\odot = \cdot_f$ and f is left e-periodic , then $f = J_e = f_e$.*
(b) *If $(X, \cdot, e, \odot)$ is a left e-magmag and $\odot = \cdot_f$, where f is left e-periodic, then X is left identical e-magmag and $f = J_e = f_e$.*

Now let's discuss more about a special case of the above topic. If $(X, \cdot, e, \odot)$ is a left e-magmag such that $\odot = \cdot_f$, for some $f : X \to X$, then $f_e = J_e$ and

$$f_e(xy) = f(e(xy)) = e \cdot_f (xy) = e \odot (xy) = J_e(xy) = J_e(x \odot y)$$
$$= J_e(x \cdot_f y) = J_e f(xy) = f(ef(xy)) = f_e f(xy) \ ; \ \forall x, y \in X.$$

The above identities give us the following relations between f and J_e
- If "$\cdot$" is surjective, then $J_e f = J_e = f_e$.
- If "$\cdot$" is surjective and f is left e-periodic, then $f = J_e$.
Therefore, if $(X, \cdot)$ is left e-unital (X is left e-unimag), then $f = J_e$.
- If e is a left identity of $f(XX)$ in $(X, \cdot)$, then:

$$J_e(xy) = J_e(x \odot y) = f(x \odot y) = J_e f(xy) = f^2(xy).$$

- If e is a left identity of $f(XX)$ in $(X, \cdot)$ and "$\cdot$" is surjective (resp. $\odot$ is surjective (i.e. $X \odot X = X$)), then $J_e = f^2 = J_e f$ (resp. $f = J_e$).
- If e is a left identity of $f(XX)$ in $(X, \cdot)$, "$\cdot$" is surjective and f is idempotent, then $f = J_e$.

Considering the previous discussion and identities, we arrive at the following important theorem that is the main key of characterization for (left) identical e-magmags.

Theorem 3.4. *Assume $(X, \cdot, e, \odot)$ is a left e-magmag such that $(X, \cdot)$ is surjective (namely left e-surmag). Then, the following statements are equivalent:*

(a) *X is left identical e-magmag;*
(b) *There exists a (unique) left e-periodic function $f : X \to X$ such that $\odot = \cdot_f$;*
(c) $\odot = \cdot_{J_e}$;
(d) *There exists a (unique) left e-periodic and idempotent function $f : X \to X$ such that $\odot = \cdot_f$;*
(e) $\odot = \odot_{J_e}$;
(g) *$f = J_e$ is the only function from X to X such that $\odot = \cdot_f$;*
(h) $\odot = \odot_{J_e} = \cdot_{J_e}$.

Proof. By applying Lemma 3.3 and the previous results, one can prove this theorem.

□

Note that if $(X, \cdot, e, \odot)$ is a (left) identical e-unimag, then the conditions of the above theorem and so all of its equivalent statements hold. Moreover, the function f does not require any additional condition. Hence, we arrive at the next corollary.

Corollary 3.5. *Suppose $(X, \cdot, e, \odot)$ is a left e-unimag. Then X is left identical e-magmag if and only if there exists a function $f : X \to X$ such that $\odot = \cdot_f$ (of course f is unique, that is the same left e-map).*

The above theorem leads us to construct left e-magmags, left identical e-magmags, and other types of them, starting with an arbitrary magma $(X, \cdot)$ and using some types of function $f : X \to X$. Also it provides a characterization of identical e-magmags.

Construction of e-magmags by f-multiplications. Suppose $(X, \cdot)$ is an arbitrary magma. Fix an element $e \in X$ and consider $f : X \to X$. Then, we have:

$$e \cdot_f (xy) = f(e(xy)) \ , \ e \cdot_f (x \cdot_f y) = f(ef(xy)).$$

Thus $(X, \cdot, e, \cdot_f)$ is a left e-magmag if and only if f satisfies the following functional equation in $(X, \cdot)$

$$f(e(xy)) = f(ef(xy)) \quad : \quad \forall x, y \in X. \tag{3.6}$$

Also $(X, \cdot, e, \cdot_f)$ is a left identical e-magmag if and only if

$$f(e(xy)) = f(ef(xy)) = f(xy) \quad : \quad \forall x, y \in X. \tag{3.7}$$

The equation (3.6) (resp. equation (3.7)) is equivalent to $f_e f|_{XX} = f_e|_{XX}$ (resp. $f_e f|_{XX} = f_e|_{XX} = f|_{XX}$). Thus if $f_e f = f_e$ (resp. $f_e f =$

$f_e = f$), then f satisfies equation (3.6) (resp. equation (3.7)), and if "$\cdot$" is surjective, then the converse is also true. Hence, if $XX = X$, then f satisfies equation (3.7) if and only if f is is left e-periodic and idempotent (equivalently $f_e = f = f^2$). Also for every $f : X \to X$, we have the following special cases :

(a) If f is right canceler, then f satisfies equation (3.6).
(b) If e is a left identity of XX in $(X, \cdot)$ and $f|_{XX}$ is idempotent, then f satisfies equation (3.7).
(c) If f is left e-periodic (e.g. if $(X, \cdot)$ is left e-unital) and f is right canceler, then f is idempotent and satisfies equation (3.7).
(d) If $(X, \cdot)$ is left e-unital, then the two functional equations (3.6), (3.7), and $f(f(x)) = f(x)$ are equivalent (so f satisfies one of them if and only if f is idempotent).

Therefore, we can construct a very vast class of left e-magmags and all left identical e-magmags (Theorem 3.6) using all functions f satisfying equations (3.6) and (3.7), some of their special cases are mentioned in the next items (I), (II), and (III). We emphasize that the e-magmags $(X, \cdot, e, \cdot_f)$ do not generate all e-magmags. If X is a set containing e and $x \odot y = x$ (for every $x, y \in X$), then for every binary operation $\cdot \neq \odot$ in X such that e is its right identity, $(X, \cdot, e, \odot)$ is a left magma-e-semigroup and $\odot \neq \cdot_f$, for all functions $f : X \to X$. Because if $\odot = \cdot_f$, then $x = x \odot y = x \cdot_f y = f(xy)$, and so $f(x) = f(xe) = x \cdot_f e = x \odot e = x$. Hence, $\cdot = \odot$, which is a contradiction.

Now let $(X, \cdot)$ be an arbitrary magma and fix $e \in X$.

(I) If f is right canceler, then $(X, \cdot, e, \cdot_f)$ is a left e-magmag (but it is not necessary that $f = J_e$).
(II) If $f : X \to X$ is left e-periodic and idempotent, then $(X, \cdot, e, \cdot_f)$ is a left identical e-magmag and $f = J_e$.
(III) If X is left e-unital and $f : X \to X$ is idempotent, then $(X, \cdot, e, \cdot_f)$ is a left identical e-unimag and $f = J_e$.

Considering the above construction, Lemma 3.3, and Theorem 3.4, we are ready to characterize all the left identical e-magmags.

Theorem 3.6. *(Characterization and construction of left identical e-magmags)*
Let $(X, \cdot)$ be a magma with the fixed element e and $\odot$ be another binary operation in X. Then, $(X, \cdot, e, \odot)$ is a left identical e-magmag if and only if there exists a (unique) left e-periodic and idempotent function f from X to X such that $\odot = \cdot_f$.
Moreover, denoting the set of all binary operations in X by $Bio(X)$,

we have:

$$\{\odot \in Bio(X) \mid (X, \cdot, e, \odot), \textit{ left identical } e\textit{-magmag}\}$$

$$= \{\cdot_f \mid f : (X, \cdot) \to (X, \cdot) \textit{ is left } e\textit{-periodic and idempotent}\}$$

$$= \{\cdot_f \mid (X, \cdot, e, \cdot_f) \textit{ is left identical } e\textit{-magmag}\}$$

$$= \{\cdot_f \mid f \textit{ satisfies equation } (3.7)\} = \{\cdot_{f_e} \mid f \textit{ satisfies equation } (3.7)\},$$

Moreover,

$$Card\{\odot \in Bio(X) \mid (X, \cdot, e, \odot) \textit{ is left identical } e\textit{-magmag}\}$$

$$= Card\{f_e \mid f \textit{ satisfies equation } (3.7)\}.$$

Proof. If $(X, \cdot, e, \odot)$ is a left identical e-magmag, then we know that $\odot = \cdot_{J_e}$ and J_e is left e-periodic and idempotent. Also if $\odot = \cdot_f$, where f is left e-periodic, then $f = f_e = J_e$ (so J_e is the only function from X to X such that $\odot = \cdot_f$).

Conversely, if there exists a left e-periodic and idempotent function f such that $\odot = \cdot_f$, then f satisfies equation (3.7), and so $(X, \cdot, e, \odot)$ is a left identical e-magmag. Note that if f satisfies equation (3.7), then $\cdot_f = \cdot_{f_e}$, and

$$(f_e)_e(y) = f(e(ey)) = f(ef(ey)) = f(ey) = f_e^2(y) = f_e(y),$$

for all $y \in X$. Thus $(f_e)_e = f_e = f_e^2$, which means that f_e is left e-periodic and idempotent. By applying these facts and the previous discussions, one can see that the above sets are equal.

Finally, define the map ψ from

$$\{\odot \in Bio(X) \mid (X, \cdot, e, \odot) \text{ is left identical } e\text{-magmag}\}$$

to

$$\{f_e \mid f \text{ satisfies equation } (3.7)\},$$

by $\psi(\odot) = f_e$ (where f is the unique function such that $\odot = \cdot_f$). It is a well-defined bijection map. □

Corollary 3.7. *Let $(X, \cdot)$ be a surjective magma with the fixed element element e and $\odot$ be another binary operation in X. Then there is a one-to-one correspondence between the set of all binary operations $\odot$ such that $(X, \cdot, e, \odot)$ is left identical e-magmag and the set of all left e-periodic and idempotent functions from X to X. Hence, if $(X, \cdot)$ is a magma with the left identity element e, then in all the results of Theorem 3.6, the term "left e-periodic and idempotent" can be replaced only by "idempotent".*

The structure of left identical e-magmags shows us more interesting and useful properties, one of which is about a relation between associativity of the second magma and a property of the e-map (as a function defined in the first magma). Since $\odot = \cdot_{J_e}$, then $(X, \odot)$ is semigroup if and only if $J_e : (X, \cdot) \to (X, \cdot)$ is associative. Thus we can drive more traits of left identical magma-e-semigroups, semigroup-e-magmas, group-e-semigroups, etc. by paying attention to the structure of such e-magmags.

Theorem 3.8. *Every left identical monoid-e-ass.magma is left (identical) monoid-e-semigroup, and e is middle identity of the (second) semigroup. Therefore, if $(X, \cdot, e, \odot)$ is a left identical e-uniassmag, then associativity of $(X, \cdot)$ implies associativity of $(X, \odot)$ (i.e. if $(X, \cdot)$ is semigroup, then $(X, \odot)$ is also semigroup).*

Proof. Put $f = J_e$. Thus $f : (X, \cdot) \to (X, \cdot)$ is canceler, by Lemma 2.7(C). Since $(X, \cdot)$ is semigroup, then f is strongly associative, and also $\odot = \cdot_f$ (by Theorem 3.6). Therefore, $(X, \odot)$ is a semigroup. Now we have:

$$x \odot e \odot y = x \cdot_f e \cdot_f y = f(f(xe)y) = f(f(x)y) = f(xy) = x \cdot_f y = x \odot y.$$

(There is also a direct proof for this lemma.) □

Since associativity of J_e plays a basic role in this part of study, we encourage to study this condition in general (in arbitrary left e-magmags). First for every $x, y, z \in X$, we have:

$$J_e(J_e(xy)z) = e\odot((e\odot(x\odot y))\odot z)\ ,\ J_e(xJ_e(yz)) = e\odot(x\odot(e\odot(y\odot z))).$$

Thus if X is left magma-e-ass.magma, then:

$$J_e(J_e(xy)z) = (e \odot e \odot (x \odot y)) \odot z\ ,\ J_e(xJ_e(yz)) = (e \odot x \odot e) \odot (y \odot z),$$

Moreover, if $e \odot x \odot e = e \odot x$ (i.e. J_e is right e-periodic or equivalently $J_e J^e = J_e$) and $e \odot e \odot (x \odot y) = e \odot (x \odot y)$ (equivalently $J_e^2|_{X \odot X} = J_e|_{X \odot X}$), then:

$$J_e(J_e(xy)z) = e \odot ((x \odot y) \odot z)\ ,\ J_e(xJ_e(yz)) = e \odot (x \odot (y \odot z)). \quad (3.8)$$

Note that if J_e is (two-sided) e-periodic or (more strongly) X is left e-uniassmag, then both conditions hold. Therefore,

J_e is associative $\Leftrightarrow$ $(X, \odot)$ is $\sim_e$ $-associative$ $\Leftrightarrow$ $(X, \cdot_{J_e})$ is semigroup,

Hence, we arrive at the following lemma as a result of the study.

Lemma 3.9. *Let X be a left e-assmag, for which one of its magmas is right $\sim_e$-unital. Then (the quotient magma exists and) the following statements are equivalent:*

(a) *J_e is associative;*
(b) *J_e is strongly associative;*
(c) *$(X, \odot)$ is $\sim_e$-associative;*
(d) *The quotient magma $X/\sim_e$ is (right $\overline{e}$-unital) semigroup;*
(e) *$(X, \cdot_{J_e})$ is semigroup.*

Proof. The above discussion, Remark 2.11 and Theorem 2.12 prove this lemma. ☐

Corollary 3.10. *Assume that X is a left identical e-assmag for which its second magma is right $\sim_e$-unital and $\sim_e$-associative. Then it is semigroup. Thus X is a left identical magma-e-semigroup if and only if J_e is associative.*

Note: Similar definitions, discussions, and constructions can be stated for the right and two-sided case. Thus $(X, \cdot, e, \odot)$ is identical e-magmag if and only if

$$e \odot xy = e \odot (x \odot y) = x \odot y = xy \odot e = (x \odot y) \odot e \;\; ; \;\; \forall x, y \in X. \quad (3.9)$$

Hence, if $(X, \cdot, e, \odot)$ is identical e-magmag, then e is a (two-sided) identity for the subset $X \odot X$ (with respect to the second binary operation).

As a subclass of the left identical e-magmags, the left identical group-e-semigroups have more useful properties (see Definition 3.16 and Lemma 3.17), and we can give the general form of f in the characterization and construction. In this class of e-magmags, the left, right, and two-sided cases are the same, as one can see in the following discussion and lemma.

Now consider a left identical e-magmag $(X, \cdot, e, \odot)$. If $(X, \cdot)$ is right e-unital, then putting $y = e$ (resp. $x = e$ and replacing y by x) in equation (3.1), we have $e \odot x = e \odot (x \odot e) = x \odot e$ (resp. $e \odot ex = e \odot (e \odot x) = e \odot x$), which implies that e is a central element of $(X, \odot)$ (resp. J_e is left e-periodic: $J_e(ex) = J_e(x)$, and so $ex \sim_e x$ for all $x \in X$). Therefore, X is (two-sided) identical e-magmag, and so equation (3.2) holds. Moreover, $(X, \odot)$ is weakly e-associative because:

$$(e \odot x) \odot e = e \odot x = x \odot e = e \odot (x \odot e)$$

(analogously, for the right identical e-magmag).

Therefore, we have proved the following lemma, where the left, right, and two-sided cases of identical structures are equivalent.

Lemma 3.11. *Consider $(X, \cdot, e, \odot)$.*

(a) *If X is left (resp. right) identical e-magmag and $(X, \cdot)$ is right (resp. left) e-unital, then X is (two-sided) identical e-wassmag and J_e is left (resp. right) e-periodic.*

(b) *If X is left or right identical e-magmag and left or right e-unimag, then X is (two-sided) identical e-uniwassmag.*

(c) *The following statements are equivalent:*

(i) *X is unital left identical e-magmag (i.e. left identical e-unimag);*

(ii) *X is unital right identical e-magmag (i.e. right identical e-unimag);*

(iii) *X is unital and identical e-magmag (i.e. identical e-unimag);*

(iv) *X is left or/and right identical e-magmag and left or/and right e-unimag.*

In addition, all the equivalent statements (i)-(iv) imply that e is a central element of $(X, \odot)$.

Therefore, construction and characterization of the left, right, and two-sided unital and identical e-magmags are the same. It will be more useful and interesting if we have an identical group-e-semigroup. Since such algebraic structures and identical e-magmags have some additional properties, we consider them in the next part of the paper.

3.1. **Magma-joined-magmas.** During the study, we see some algebraic structures $(X, \cdot, \odot)$ such that the e-join law holds, for every $e \in X$. They have close relations to identical e-magmags if both magmas are semigroups.

Definition 3.12. Let X be a set with the two binary operations "$\cdot$" and $\odot$. We call $(X, \cdot, \odot)$ *left magma-joined-magma* or *left joined-magmag* (briefly) if

$$t \odot xy = t \odot (x \odot y) \;\; ; \;\; \forall t, x, y \in X \;\; \text{(left join law)}. \tag{3.10}$$

Therefore, $(X, \cdot, \odot)$ is left joined-magmag if and only if it is left e-magmag for every $e \in X$.

Example 3.13. The algebraic structure $(\mathbb{R}, +, +_b)$ is group-joined-semigroup (b-real group-grouplike), because

$$t +_b (x + y) = (t + x + y)_b = (t + (x + y)_b)_b = t +_b x +_b y.$$

The Klein group-grouplike is a finite group-joined-semigroup. Also the algebraic structures $(X, \cdot, \odot)$ in Example 2.2(a) and Example 2.2(c) are (trivial) joined-magmag and magma-joined-semigroup, respectively.

It is interesting to know that every left identical semigroup-e-semigroup is semigroup-joined-semigroup.

Lemma 3.14. *Let S be a set with the fixed element e and two binary operations "$\cdot$" and $\odot$. Then:*

$$(S, \cdot, e, \odot) \textit{ is left identical semigroup-e-semigroup}$$

$$\Rightarrow (S, \cdot, \odot) \text{ is left semigroup-joined-semigroup}$$
$$\Rightarrow \ (S, \cdot, e, \odot) \text{ is left semigroup-e-semigroup.}$$

But the converses are not valid. Therefore, if there exists $e_0 \in S$ *such that* $(S, \cdot, e_0, \odot)$ *is left identical semigroup-*e_0*-semigroup, then* $(S, \cdot, e, \odot)$ *is left semigroup-e-semigroup for every* $e \in S$.

Proof. If $(S, \cdot, e, \odot)$ is left identical semigroup-e-semigroup and $t, x, y \in X$, then:

$$t \odot xy = e \odot (t \odot xy) = e \odot (txy) = e \odot (t \odot x \odot y) = t \odot x \odot y.$$

Thus the first part of the lemma is proved.

Now consider a semigroup $(S, \odot)$ containing the left zero 0_ℓ and the left identity 1_ℓ. If $\cdot \neq \odot$ is another associative binary operation in S, then $(S, \cdot, 0_\ell, \odot)$ is left semigroup-0_ℓ-semigroup. But it is not left semigroup-1_ℓ-semigroup and left semigroup-joined-semigroup (e.g. $([0, +\infty), +, 0, \cdot)$, where $+$ and $\cdot$ are the usual addition and multiplication).

Also, if $(S, \cdot)$ is an arbitrary semigroup with more than two elements and $x \odot y = x$, for every $x, y \in S$, then $(S, \cdot, \odot)$ is a left semigroup-joined -semigroup. But, if e is an arbitrary fixed element of S, then $e \odot (xy) = e = e \odot x \odot y \neq x \odot y = x$ (for every $x \neq e$). Thus $(S, \cdot, e, \odot)$ is not left identical semigroup- e -semigroup.

□

3.2. **More properties for (two-sided) e-magmags.** The(two-sided) e-magmags have more properties than the left and right cases, and there is a type of them between e-magmags and identical e-magmags. In this section, we focus on them and introduce another class of (two-sided) e-magmags that has so close relation to the identical e-magmags and magma-joined-magmas.

By $Z(X)$ (resp. $Zt(X)$), we denote the set of all central [rep. central idempotent] elements of X.

(A) Suppose $(X, \cdot, e, \odot)$ **is an** **$\underline{e\text{-magmag}}$**:

- If e is an idempotent element of $(X, \cdot)$ (i.e. $e^2 = e$), then $e^{\odot 3}$ is well-defined, and $e^{\odot 3} = e^{\odot 2}$.
- If $(X, \cdot)$ is left (resp. right) e-unital, then $e \odot x = e \odot (e \odot x)$, $x \odot e = (e \odot x) \odot e$ (resp. $e \odot x = e \odot (x \odot e)$ and $x \odot e = (x \odot e) \odot e$) for every $x \in X$. Hence, if e is its identity, then:

$$e \odot x = e \odot (e \odot x) = e \odot (x \odot e) \ , \ x \odot e = (x \odot e) \odot e = (e \odot x) \odot e \quad (3.11)$$

- Therefore, if $(X, \cdot)$ is left e-unital, then:

$$e \in Z(X, \odot) \Leftrightarrow e \text{ is a right identity of } e \odot X \text{ in } (X, \odot)$$

$$\Leftrightarrow e \text{ commutes with every element of } e \odot X \text{ in } (X, \odot)$$
$$\Leftrightarrow e \text{ is an identity of } e \odot X \text{ in } (X, \odot)$$

(analogously, if e is a right identity).

Hence, if e is its identity, then:

$$e \in Z(X, \odot) \Leftrightarrow e \text{ is an identity of } e \odot X \text{ in } (X, \odot)$$
$$\Leftrightarrow (X, \odot) \text{ is weakly } e\text{-associative}$$
$$\Leftrightarrow e \text{ commutes with every elements of } e \odot X \text{ in } (X, \odot)$$
$$\Leftrightarrow e \odot (e \odot x) = (x \odot e) \odot e$$
$$\Leftrightarrow e \odot x = e \odot (e \odot x) = e \odot (x \odot e) = x \odot e = (e \odot x) \odot e = (x \odot e) \odot e,$$

for all $x \in X$. Thus we have proved the following lemma.

Lemma 3.15. *In every e-uniwassmag, e is the central element of the second magma. Hence, an e-unimag is e-uniwassmag if and only if e is second-central.*

Now we need to introduce other classes of e-magmags that are between e-magmags and identical e-magmags.

Definition 3.16. We call an e-magmag $(X, \cdot, e, \odot)$ *full e-magmag* if

$$e \odot xy = e \odot (x \odot y) = xy \odot e = (x \odot y) \odot e \;\; ; \;\; \forall x, y \in X. \quad (3.12)$$

Note that X satisfies the above identities (3.12) if and only if it satisfies:

$$t \odot xy = t \odot (x \odot y) = xy \odot t = (x \odot y) \odot t \;\; ; \;\; t, x, y \in P,$$

for (only) $t = e$, that is our idea for introducing "*full joined-magmags*".

At the first, we have the following elementary properties about them:

(i) The following implications hold, but the converses are not true;

$$X \text{ is an identical } e\text{-magmag} \Rightarrow X \text{ is a full } e\text{-magmag}$$
$$\Rightarrow X \text{ is an } e\text{-magmag}.$$

We only give counter-examples; the implications are obvious.

X is an e-magmag $\not\Rightarrow$ X is a full e-magmag : consider an arbitrary trivial e-magmag $(X, \cdot, e, \cdot)$, for which e does not commute with some elements of XX.

X is a full e-magmag $\not\Rightarrow$ X is an identical e-magmag : consider an arbitrary e-magmag $(X, \cdot, e, \odot)$, for which $(X, \odot)$ is a magma with the zero $e = 0$, and it is not null (i.e. $x_0 y_0 \neq e$ for some $x_0, y_0 \in X$).

(ii) Every left (right) e-magmag is a full e-magmag if and only if e commutes with every element of $X \odot X \cup XX$ in $(X, \odot)$ (e.g. if e is second-central).

(iii) An e-magmag $(X, \cdot, e, \odot)$ is full if and only if one (all) of the following conditions hold:
(a) e commutes with every element of $X \odot X$ in $(X, \odot)$,
(b) e commutes with every element of XX in $(X, \odot)$,
(iv) A full e-magmag $(X, \cdot, e, \odot)$ is identical if and only if one (all) of the following conditions hold:
(a) e is a left or right identity of $X \odot X$ in $(X, \odot)$,
(b) e is an identity of $X \odot X$ in $(X, \odot)$.

Now we arrive at the following lemma.

Lemma 3.17. *If one of the following conditions holds, then e is second-central (and then X is a full e-magmag):*

(1) *X is a left e-magmag such that $(X, \cdot)$ is right e-unital and e is a left identity of $X \odot e$ in $(X, \odot)$.*
(2) *X is a right e-magmag such that $(X, \cdot)$ is left e-unital, and e is a right identity of $e \odot X$ in $(X, \odot)$.*
(3) *X is an e-magmag, and one of the following properties holds:*
(3-1) *$(X, \cdot)$ is left e-unital, and e commutes with every elements of $e \odot X$ in $(X, \odot)$.*

(3-2) *$(X, \cdot)$ is right e-unital, and e commutes with every element of $X \odot e$ in $(X, \odot)$.*

(4) *X is an e-unimag, and e commutes with every elements of $e \odot X$ or $X \odot e$ in $(X, \odot)$.*

Proof. If (1) holds, then:

$$e \odot x = e \odot (xe) = e \odot (x \odot e) = x \odot e,$$

and similarly for (2). Now one can get the results by the previous explanations. □

Similar to the above discussion, we have more properties if $(X, \cdot, e, \odot)$ is a full e-magmag.

(B) Suppose that $(X, \cdot, e, \odot)$ is a full e-magmag:

- If $(X, \cdot)$ is left (resp. right) e-unital, then $e \odot x = e \odot (e \odot x) = x \odot e = (e \odot x) \odot e = xe \odot e$ (resp. $e \odot x = e \odot (x \odot e) = x \odot e = (x \odot e) \odot e = ex \odot e$) for every $x \in X$. Thus $e \in Z(X, \odot)$, $(X, \odot)$ is weakly e-associative, e is an identity of $e \odot X = X \odot e$ in $(X, \odot)$, J_e is idempotent, and also J_e is e-periodic with respect the two binary operations, i.e:

$$J_e(e \odot x) = J_e(x \odot e) = J_e(x) = J_e(ex) = J_e(xe) \ (= J_e^2(x)).$$

- If $(X, \cdot)$ is e-unital, then we have all the above properties, and X is an e-uniwassmag.

(C) Finally, assume that $(X,\cdot,e,\odot)$ is an identical e-magmag: Without any additional condition, we have:

$$\begin{aligned} e\odot ex &= e\odot(e\odot x)=e\odot x=(e\odot x)\odot e=ex\odot e\\ e\odot xe &= e\odot(x\odot e)=x\odot e=(x\odot e)\odot e=xe\odot e. \end{aligned} \tag{3.13}$$

Thus:

$$J^e(e\odot x)=J^eJ_e(x)=J_e(e\odot x)=J_e^2(x)=J_e(x)=J^e(ex)=J_e(ex);$$

$$J_e(x\odot e)=J_eJ^e(x)=J^e(x\odot e)=J^{e^2}(x)=J^e(x)=J_e(xe)=J^e(xe).$$

Therefore, J_e and J^e are idempotent maps and J_e (resp. J^e) is left (resp. right) e-periodic with respect to two binary operations.
- We have the following equivalent conditions:

$$e\in Z(X,\odot)\Leftrightarrow (X,\odot)\text{ is weakly } e\text{-associative}$$

$$\Leftrightarrow e\odot(e\odot x)=(x\odot e)\odot e\ :\ \forall x\in X\Leftrightarrow ex\odot e=xe\odot e\ ;\ \forall x\in X$$

$$\Leftrightarrow e\odot ex=e\odot x=ex\odot e=e\odot xe=x\odot e=xe\odot e\ ;\ \forall x\in X$$

Thus if $(X,\cdot)$ is left or right e-unital, then all of the above properties hold.
- If $(X,\cdot)$ is e-unital, then X is identical e-uniwassmag, and e and $e\odot e$ are central elements of $(X,\odot)$. Moreover, if $(e\odot e)\odot(e\odot e)=e\odot(e\odot(e\odot e))$ or $(e\odot e)\odot(e\odot e)=((e\odot e)\odot e)\odot e$, then $e\odot e\in Zt(X,\odot)\neq\emptyset$, because:

$$(e\odot e)\odot(e\odot e)=e\odot(e\odot(e\odot e))=e\odot(e\odot e)=e\odot e.$$

Hence, if X is an identical e-uniassmag, then $e\odot e\in Zt(X,\odot)$ and $e\in Z(X,\odot)$.

4. Relations to Grouplikes

One of the most important properties of a class of e-magmags is that their second magma has a unique central idempotent element. The property is more useful and interesting if it is an identical loop-e-semigroup. If this is the case, the second magma is a *grouplike*. Hence, at first, we give a summary about grouplikes introduced in [4].

A grouplike is a semigroup $(\Gamma,\cdot)$ that satisfies the following axioms:

(1) There exists $\varepsilon\in\Gamma$ such that:

$$\varepsilon x=\varepsilon^2x=x\varepsilon^2=x\varepsilon\ :\ \forall x\in\Gamma,$$

(2) For every ε satisfying (1) and every $x\in\Gamma$, there exists $y\in\Gamma$ such that:

$$xy=yx=\varepsilon^2.$$

Every $\varepsilon \in \Gamma$ satisfying the axioms (1) and (2) is called an *identity-like*.

As one can see in [4], condition (1) is equivalent to $Zt(\Gamma) \neq \emptyset$, and every grouplike contains a unique idempotent identity-like element that is also its unique central idempotent.

Now, let Γ be a grouplike and let ϵ be the unique idempotent identity-like element of Γ. Then, we call ϵ *standard identity-like*, and use the notation $(\Gamma, \cdot, \epsilon)$. Γ is a *standard grouplike* if ϵ is the only idempotent of Γ. Γ is a *zero grouplike* if ϵ is a zero of Γ. Every y that corresponds to x in axiom (2) is called *inverse-like of* x, and is denoted by x'_ϵ or x'. If $(\Gamma, \cdot)$ is grouplike and is not group, then we call it *proper grouplike*. Now, we state a clear comparison between the grouplike and group axioms:

Lemma 4.1. *A structure* $(\Gamma, \cdot)$ *is a grouplike if and only if it satisfies the following axioms:*

(i) *Closure;*
(ii) *Associativity;*
(iii) *There exists a* <u>*unique*</u> *element* $e \in \Gamma$ *such that* $ex = xe$, *and* $e^2 = e$ *for all* $\overline{x \in X}$*;*
(iv) *For every* $x \in \Gamma$*, there exists* $y \in \Gamma$ *(not necessarily unique) such that:*

$$xy = yx = e.$$

Proof. See [4]. □

Corollary 4.2. *A semigroup* $(\Gamma, \cdot)$ *is grouplike if and only if* $Zt(\Gamma) = \{e\}$*, and for every* $x \in \Gamma$*, there exists* $y \in \Gamma$ *such that* $xy = yx = e$*.*

Example 4.3. Every group is standard group-like.

The structure $(\mathbb{R}, +_b, 0)$ is a proper grouplike (namely *real b-grouplike*, and specially *real grouplike* if $b = 1$). The set of all identity-likes of $(\mathbb{R}, +_b)$ is $b\mathbb{Z}$ that is also the set of all inverse-likes of 0.

Every semigroup S containing the zero for which 0 is its only central idempotent element is a zero grouplike.

The magma $(K, \odot)$ is Klein four-grouplike, and $\{e, \eta\}$ is the set of all its identity-likes.

Now we come back to our topic and show an important property of magma-e-semigroups related to grouplikes.

Theorem 4.4. *Let* $(X, \cdot, e, \odot)$ *be a unital magma-e-semigroup. Then:*

(a) $e \in Z(X, \odot)$*,* $e \odot e \in Zt(X, \odot)$*, and* $(e \odot X, \odot, e \odot e)$ *is a monoid.*

(b) *If e is a left (resp. right) identity of the sub-semigroup $Zt(X,\odot)$ and the equation $x\delta = e$ (resp. $\delta x = e$) has a solution in X for every $\delta \in Zt(X,\odot)$, then $Zt(X,\odot) = \{e \odot e\}$.*

(c) *If e is a left (resp. right) identity of $It(X,\odot)$ and the equation $x\delta = e$ (resp. $\delta x = e$) has a solution in X, for every $\delta \in It(X,\odot)$, then $It(X,\odot) = Zt(X,\odot) = \{e \odot e\}$.*

(d) *If e is a left (esp. right) identity of $Zt(X,\odot)$ and the equation $xy = e$ (resp. $yx = e$) has a solution in X, for every $y \in X$, then $(X,\odot,e\odot e)$ is a grouplike (and so $(e\odot X,\odot,e\odot e)$ is a group).*

Proof. Let $\delta \in It(X,\cdot)$, $e \odot \delta = \delta$, and $\beta\delta = e$ for some $\beta \in X$. Then:

$$e \odot \beta \odot \delta = e \odot \beta\delta = e \odot e,$$

and so:

$$\delta = e \odot \delta = e \odot e\delta = e \odot e \odot \delta = e \odot \beta \odot \delta \odot \delta = e \odot \beta \odot \delta = e \odot e$$

Now Lemma 3.17, Corollary 4.2, and part (C) at the end of the pervious section complete the proof. □

Future direction of the research. An important research field of the topic is the study of the left, right, two-sided, and full magma-joined-magmas, specially group-joined-semigroups, loop-joined-semigroups, and their characterization and comparison to the magma-distributed-magmas (specially ring-like structures). Another one study of grouplikes as the second magma of identical group-e-semigroups. Also it is connected to the topic functional equations on algebraic structures and all magmas with a single binary operation.

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MAGMA-JOINED-MAGMAS: A CLASS OF NEW ALGEBRAIC STRUCTURES

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ماگما-پیوست-ماگما : دسته‌ای از ساختارهای جبری جدید

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منظور ما از ماگما -e-ماگما مجموعه‌ای شامل عضو ثابت e و مجهز به عمل دوتایی "$\cdot$" و $\odot$ با ویژگی $e\odot(x\cdot y)=e\odot(x\odot y)$ به نام قانون e-وصل چپ می‌باشد. بنابراین $(X,\cdot,e,\odot)$ یک ماگما -e-ماگما چپ است اگر و تنها اگر $(X,\cdot)$ و $(X,\odot)$ ماگما (گروه‌واره) باشند، $e\in X$ و قانون e-وصل چپ برقرار باشد. ماگما -e-ماگما راست و دوطرفه نیز به روش مشابه تعریف می‌شود. همچنین X ماگما -پیوست-ماگما است اگر و تنها اگر ماگما -x-ماگما باشد برای هر $x\in X$. بنابراین ما یک دسته بزرگی از ساختارهای جبری با دو عمل دوتایی معرفی می‌کنیم که بعضی از زیر کلاس های آنها گروه -e-نیم‌گروه ، دور -e-نیم‌گروه ، نیم‌گروه -e-گروه‌واره و غیره می‌باشد. یک مثال زیبای نامتناهی (متناهی) برای آنها گروه-شبه‌گروه حقیقی $(\mathbb{R},+,\circ,+_1)$ (گروه-شبه‌گروه کلاین) می‌باشد. در این مقاله ما آن مبحث را معرفی می‌کنیم، چندین دسته بزرگ از چنین ساختارهای جبری را ساخته و همه ماگما -e-ماگماهای یکانی را با چند روش مشخص‌سازی می‌نماییم. انگیزه این مطالعه به خاطر بعضی از ارتباط‌ها با f-ضرب‌ها، بعضی از معادلات تابعی روی ساختارهای جبری و شبه‌گروه‌ها (که به تازگی توسط نویسنده معرفی شده‌است) می‌باشد. سرانجام ما بعضی از افق‌های تحقیقیاتی در این موضوع را شرح می‌دهیم.

کلمات کلیدی: ساختار جبری، شبه‌گروه، ماگما .