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NONNIL-NOETHERIAN MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. In this paper, we introduce a new class of modules that is closely related to the class of Noetherian modules. Let R be a commutative ring with identity, and M be an R-module such that $Nil(M)$ is a divided prime submodule of M. M is called a nonnil-Noetherian R-module if every nonnil submodule of M is finitely-generated. We prove that many properties of the Noetherian modules are also true for the nonnil-Noetherian modules.

Throughout this paper, all rings are commutative with $1 \neq 0$, and all modules are unitary. Let R be a commutative ring with identity, and $Nil(R)$ be the set of nilpotent elements of R. Recall from [17] and [9] that a prime ideal of R is called a *divided prime ideal* if $P \subset Rx$ for every $x \in R \setminus P$. Thus a divided prime ideal is comparable to every ideal of R. In [9], [10], [11], [12], [13], and [14] shown that the class of rings, $\mathcal{H} = \{R|R$ is a commutative ring, and that $Nil(R)$ is a divided prime ideal of $R\}$. In [7] and [8], Anderson and Badawi have generalized the concepts of Prüfer, Dedekind, Krull, and Bezout domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and Badawi [15] have generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. Let R be a ring, $Z(R)$ be the set of zero-divisors of R, and $S = R \setminus Z(R)$. Then $T(R) := S^{-1}R$ denotes the total quotient ring of R. We start by recalling some background materials. A non-zero-divisor of a ring R is called a *regular element,*

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and an ideal of R is said to be regular if it contains a regular element. An ideal I of a ring R is said to be a *nonnil ideal* if $I \nsubseteq Nil(R)$. If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then $Nil(R) \subset I$. In particular, this holds if I is a regular ideal of a ring $R \in \mathcal{H}$. Recall from [10] that for a ring $R \in \mathcal{H}$, the map $\phi : T(R) \to R_{Nil(R)}$, given by $\phi((a/b) = a/b$, for $a \in R$ and $b \in R \setminus Z(R)$, is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$, given by $\phi(x) = x/1$ for every $x \in R$.

Let R be a ring, and M be an R-module. M is called a *cancellation module* if whenever $IM = JM$ for ideals I and J of R, then $I = J$ (see [20]). For a submodule N of M, we denote by $(N :_R M)$ the residual of N by M, i.e. the set of all $r \in R$ such that $rM \subseteq N$. The annihilator of M, which is denoted by $ann_R(M)$, is then $(0 :_R M)$. An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Note that since $I \subseteq (N :_R M)$, then $N = IM \subseteq (N :_R M)M \subseteq N$ so that $N = (N :_R M)M$ [22]. Finitely-generated faithful multiplication modules are cancellation modules [22, Theorem 3.1]. For a submodule N of M, if $N = IM$ for some ideal I of R, then we say that I is a presentation ideal of N. Note that it is possible that for a submodule N, no such presentation ideal exists. For example, assume that M is a vector space over an arbitrary field F with $dim_F M \geq 2$, and let N be a proper subspace of M such that $N \neq 0$. Then if N has a presentation ideal, then $N = IM$ for some ideal I of F. Since the only ideals of F are 0 and F itself, $I = 0$ or $I = F$. Hence, $N = 0$ or $N = M$, a contradiction. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be the submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K, denoted by NK, is defined by $NK = I_1I_2M$. Then, by [5, Theorem 3.4], the product of N and K is independent from presentations of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [5]).

Let R be a ring, and M an R-module. An element $r \in R$ is called a *zero-divisor* on M, provided that $rm = 0$ for some non-zero $m \in M$. We denote by $Z_R(M)$ (briefly, $Z(M)$) the set of all zero-divisors of M. It is easy to see that $Z(M)$ is not necessarily an ideal of R but it has the property that if $a, b \in R$ with $ab \in Z(M)$, then either $a \in Z(M)$ or $b \in Z(M)$. A submodule N of M is called a *nilpotent submodule* if $(N :_R M)^n N = 0$ for some positive integer n. An element $m \in M$ is said to be nilpotent if Rm is a nilpotent submodule of M [3]. We let $Nil(M)$ to denote the set of all nilpotent elements of M. Then $Nil(M)$

is a submodule of M, provided that M is a faithful module, and if, in addition, M is multiplication, then $Nil(M) = Nil(R)M = \bigcap P$, where the intersection runs over all prime submodules of M, [3, Theorem 6]. If M contains no non-zero nilpotent elements, then M is called a reduced R-module. A submodule N of M is said to be a *nonnil submodule* if $N \nsubseteq Nil(M)$. We recall that a proper submodule N of M is *prime* if, for every $r \in R$ and $m \in M$ with $rm \in N$, either $m \in N$ or $rM \subseteq N$. If N is a prime submodule of M, then $p := (N :_R M)$ is a prime ideal of R. In this case, we say that N is a p-prime submodule of M. Let N be a submodule of a multiplication R-module M. Then N is a prime submodule of M if and only if $(N :_R M)$ is a prime ideal of R if and only if $N = pM$ for some prime ideal p of R with $(0 :_R M) \subseteq p$, [22, Corollary 2.11]. We recall from [4] that a prime submodule of M is called a divided prime submodule of M if $P \subset Rm$ for every $m \in M \setminus P$. Thus a divided prime submodule is comparable to every submodule of M.

Let M be an R-module, and set

$$T = \{t \in S : \text{for all m} \in M, \text{ with tm} = 0, \text{m} = 0\} = (R \setminus Z(M)) \cap (R \setminus Z(R)).$$

T is a multiplicatively-closed subset of S, and if M is torsion-free, then $T = S$. In particular, if M is a faithful multiplication R-module, then $T = S$ [22, Lemma 4.1]. We denote $T^{-1}M$ by $\mathfrak{T}(M)$.

Let R be a commutative ring, and set

$$\mathbb{H}(R) = \{M | M \text{ is an } R\text{-module, and } Nil(M) \text{ is a divided prime submodule of } M\},$$

and

$$\mathbb{H}_0(R) = \{M \in \mathbb{H} | Nil(M) = Z(M)M\}.$$

If $M \in \mathbb{H}(R)$ (resp., $M \in \mathbb{H}_0(R)$), then we may write $M \in \mathbb{H}$ (resp., $M \in \mathbb{H}_0$) instead if there is no confusion. For an R-module $M \in \mathbb{H}$, $Nil(M)$ is a prime submodule of M. Thus $P := (Nil(M) :_R M)$ is a prime ideal of R.

Lemma 1. *Let R be a commutative ring, and M an R-module with $Nil(M)$, a proper submodule. Then, $(Nil(M) :_R M) \subseteq Z(M)$.*

Proof. If $(Nil(M) :_R M) \nsubseteq Z(M)$, then, there exists $a \in R \setminus Z(M)$ with $a \in (Nil(M) :_R M)$. As $Nil(M)$ is a proper submodule of M, there exists $m \in M \setminus Nil(M)$. In this case, $am \in Nil(M)$. Thus there exists a positive integer k such that $(Ram :_R M)^k Ram = 0$. Then we have $((Ram :_R M)^k Rm)a = (Ram :_R M)^k Ram = 0$. As $a \notin Z(M)$, we have $(Ram :_R M)^k Rm = 0$. On the other hand, $a^k(Rm :_R M)^k Rm \subseteq (Ram :_R M)^k Rm = 0$. Moreover, since $a \notin Z(M)$, $a^k \notin$

$Z(M)$. Thus $(Rm :_R M)^k Rm = 0$ which, means that $m \in Nil(M)$, a contradiction. □

Let R be a commutative ring, and M an R-module with $Nil(M)$ a proper submodule. By Lemma 1, $R \setminus Z(M) \subseteq R \setminus (Nil(M) :_R M)$. In particular, $T \subseteq R \setminus (Nil(M) :_R M)$. Thus we can define a mapping $\Phi : \mathfrak{T}(M) \to M_P$, given by $\Phi(x/s) = x/s$, which is clearly an R-module homomorphism. The restriction of Φ to M is also an R-module homomorphism from M into M_P given by $\Phi(m) = m/1$ for every $m \in M$.

Badawi [14] defined a commutative ring R to be a nonnil-Noetherian ring if every nonnil ideal of R is finitely-generated. In this paper, we introduce a generalization of nonnil-Noetherian rings. Let R be a commutative ring. An R-module M is called a nonnil-Noetherian module if every nonnil submodule of M is finitely-generated. We study the basic properties of the nonnil-Noetherian modules. Moreover, we study the interplay between the nonnil-Noetherian rings and the nonnil-Noetherian modules.

Proposition 2. *Let R be a commutative ring, and M a finitely-generated faithful multiplication R-module. Then $Nil(R) = (Nil(M) :_R M)$.*

Proof. Since M is faithful, $Nil(M)$ is a submodule of M by [3, Theorem 6]. Therefore $Nil(M) = (Nil(M) :_R M)M$ since M is a multiplication module. On the other hand, since M is a faithful multiplication R-module, it follows from [3, Theorem 6] that $Nil(M) = Nil(R)M$. Furthermore, by [22, Theorem 3.1], M is a cencellation R-module. Consequently, $Nil(R) = (Nil(M) :_R M)$. □

Proposition 3. *Let R be a commutative ring, and M a finitely-generated faithful multiplication R-module. Then $Nil(M)_q = Nil(M_q)$ for every prime ideal q of R.*

Proof. Since M is a finitely-generated faithful multiplication R-module, M_q is a finitely-generated multiplication R_q-module by [21, Lemma 9.12] and [6, Corollary 3.5]. Moreover, since M is finitely-generated, we have $(0 :_{R_q} M_q) = (0 :_R M)_q = 0$, i.e. M_q is a faithful R_q-module. Hence, by [3, Theorem 6], we have:

$$Nil(M)_q = [Nil(R)M]_q = Nil(R)_q M_q = Nil(R_q)M_q = Nil(M_q).$$

□

Let R be a commutative ring. We define $\mathcal{H}_0$ as follows:

$$\mathcal{H}_0 = \{R \in \mathcal{H} | Nil(R) = Z(R)\}.$$

Proposition 4. *Let R be a commutative ring, and M be a finitely-generated faithful multiplication R-module.*

(1) *$R \in \mathcal{H}$ if and only if $M \in \mathbb{H}$.*
(2) *$R \in \mathcal{H}_0$ if and only if $M \in \mathbb{H}_0$.*

Proof. (1) $R \in \mathcal{H}$ if and only if $Nil(R)$ is a divided prime ideal of R if and only if $(Nil(M) :_R M)$ is a divided prime ideal of R by Proposition 2, if and only if $Nil(M)$ is a divided prime submodule of M by [4, Proposition 6], if and only if $M \in \mathbb{H}$.

(2) First note that since M is a faithful multiplication R-module, it is torsion-free, by [22, Lemma 4.1]. Thus $T = S$, which implies that $Z(M) \subseteq Z(R)$. On the other hand, we have $Z(R) \subseteq Z(M)$ since M is faithful. Hence, $Z(R) = Z(M)$. Now $R \in \mathcal{H}_0$ if and only if $Nil(R) = Z(R)$ if and only if $Nil(R)M = Z(R)M = Z(M)M$ if and only if $Nil(M) = Z(M)M$ by [3, Theorem 6] if and and only if $M \in \mathcal{H}_0$. □

Proposition 5. *Let R be a commutative ring, and q a prime ideal of R. If M is a finitely-generated faithful multiplication R-module with $M \in \mathbb{H}(R)$, then $M_q \in \mathbb{H}(R_q)$.*

Proof. Since q is a prime ideal of R and M a finitely-generated faithful multiplication R-module, it follows from [22, Corollary 2.11] that qM is a prime submodule of M. Hence $Nil(M) \subseteq qM$ by [3, Theorem 6]. Hence, $(Nil(M) :_R M)M \subseteq qM$, and since M is a cancellation R-module, we have $(R \setminus q) \cap (Nil(M) :_R M) = \emptyset$. Therefore, by Proposition 3, $Nil(M_q) = Nil(M)_q$ is a prime submodule of M_q. Now suppose that $m = x/s \notin Nil(M_q)$. Then $x \notin Nil(M)$ and $Nil(M)$ divided prime gives $Nil(M) \subset Rx$. If $a/t \in Nil(M_q) = Nil(M)_q$, then $a \in Nil(M) \subset Rx$. Thus $a = rx$ for some $r \in R$. In this case, $a/t = (rx)/t = (srx)/(st) = ((sr)/t)m \in R_q m$, i.e. $Nil(M_q) \subset R_q m$. Therefore, $Nil(M_q)$ is a divided prime submodule of M_q, and hence, $M_q \in \mathbb{H}(R_q)$. □

Theorem 6. *(*[19, Theorem 5]*) A non-zero finitely-generated R-module M is Noetherian if and only if every prime submodule of M is finitely generated.*

Lemma 7. *(*[23, Lemma 2.5] *Let R be a ring, and M a finitely-generated faithful multiplication R-module such that $M \in \mathbb{H}$. Then $M/Nil(M)$ is isomorphic to $\Phi(M)/Nil(\Phi(M))$ as R-modules.*

Theorem 8. *Let R be a commutative ring, and let $M \in \mathbb{H}$ be an R-module. The following statements are equivalent:*

(1) *M is a nonnil-Noetherian R-module.*

(2) *For every nonnil submodule N of M, M/N is a Noetherian R-module.*
(3) *M satisfies ACC on nonnil submodules.*
(4) *M satisfies ACC on nonnil finitely-generated submodules.*

Proof. (1) $\Rightarrow$ (2) Let M be a nonnil-Noetherian R-module. Suppose that N is a nonnil submodule of M. Let K/N be a non-zero submodule of M/N. Then K is a nonnil submodule of M. Since M is nonnil-Noetherian, K is finitely-generated, and so K/N is finitely-generated. Hence, M/N is a Noetherian R-module.

(2) $\Rightarrow$ (3) Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of nonnil submodules of M. In this case, M/N_1 is a Noetherian R-module by assumption. Moreover, $N_2/N_1 \subseteq N_3/N_1 \subseteq \cdots$ is an ascending chain of submodules of M/N_1. Since M/N_1 is Noetherian, there exists a positive integer t such that $N_t/N_1 = N_s/N_1$ for every $s \geq t$. Thus $N_t = N_s$ for every $s \geq t$.

(3) $\Rightarrow$ (4) Is clear.

(4) $\Rightarrow$ (1) If M is not a nonnil-Noetherian R-module, then there exists a nonnil submodule N of M such that N is not finitely-generated. Choose a non-nilpotent element $m_1 \in N$. Then $Rm_1 \subseteq N$, and since N is not finitely-generated, $N \neq Rm_1$. Now choose a non-zero element $m_2 \in N \setminus Rm_2$. In this case, $Rm_1 + Rm_2 \subset N$. Thus we can choose a non-zero $m_3 \in N \setminus (Rm_1 + Rm_2)$. Then $Rm_1 + Rm_2 + Rm_3 \subset N$. Continuing this way, we get a strictly ascending chain $Rm_1 \subset Rm_1 + Rm_2 \subset Rm_1 + Rm_2 + Rm_3 \subset \cdots$ of nonnil submodules of M, a contradiction. Thus M is a nonnil-Noetherian R-module. $\square$

Theorem 9. *Let R be a commutative ring, and M be an R-module such that $Nil(M)$ is a submodule of M. If M is a nonnil-Noetherian R-module, then $M/Nil(M)$ is a Noetherian R-module. The converse is true if $M \in \mathbb{H}$.*

Proof. Assume that M is a nonnil-Noetherian R-module. Set $L = M/Nil(M)$, and let Q be a non-zero prime submodule of L. Then $Q = P/Nil(M)$ for some nonnil prime submodule P of M, and hence, P is finitely-generated. It obviously follows that $Q = P/Nil(M)$ is a finitely-generated submodule of L. Hence, L is a Noetherian R-module by [19, Theorem 5]. Conversely, suppose that $M/Nil(M)$ is Noetherian, and $M \in \mathbb{H}$. If N is a nonnil submodule of M, then if follows from $M \in \mathbb{H}$ that $Nil(M) \subseteq N$, and hence:

$$\frac{M}{N} \cong \frac{\frac{M}{Nil(M)}}{\frac{N}{Nil(M)}}$$

is Noetherian. Thus M satisfies condition (2) of Theorem 8 and is nonnil-Noetherian. □

Corollary 10. *Let R be a commutative ring, and M an R-module with $M \in \mathbb{H}$. If every nonnil prime submodule of M is finitely-generated, then M is a nonnil-Noetherian R-module.*

Proof. Suppose that every nonnil prime submodule of M is finitely-generated. Then every (nonzero) prime submodule of $L = M/Nil(M)$ is finitely-generated. Hence, L is a R-module by Theorem 6. Thus, M is a nonnil-Noetherian R-module by Theorem 9. □

Proposition 11. *(*[23, Proposition 2.2] *Let R be a commutative ring, and M a finitely-generated faithful multiplication R-module with $M \in \mathbb{H}$. Then $\Phi(M) \in \mathbb{H}$.*

Corollary 12. *Let R be a commutative ring and M an R-module with $M \in \mathbb{H}$. The following statements are equivalent:*

(1) *M is a nonnil-Noetherian R-module.*
(2) *$M/Nil(M)$ is a Noetherian R-module.*
(3) *$\Phi(M)/Nil(\Phi(M))$ is a Noetherian R-module.*
(4) *$\Phi(M)$ is a nonnil-Noetherian R-module.*

Proof. (1) ⇒ (2) This follows from Theorem 9. (2) ⇒ (3) This is a direct consequence of Lemma 7. (3) ⇒ (4) Again follows from Theorem 9 because $\Phi(M) \in \mathbb{H}$ by Proposition 11. □

Theorem 13. *Let R be a commuative ring, and M a finitely-generated multiplication R-module. Then M is a nonnil-Noetherian R-module if and only if R is a nonnil-Noetherian ring.*

Proof. Assume that M is a nonnil-Noetherian R-module, and let I be a nonnil ideal of R. Then IM is a nonnil submodule of M by Proposition 2. Hence, IM is finitely-generated submodule of M. It follows from the fact that M is a cancellation R-module and [16, Lemma 3.5] that I is a finitely-generated ideal of R. Consequently, R is a nonnil-Noetherian ring. Conversely, assume that R is a nonnil-Noetherian ring, and let N be a nonnil submodule of M. Then by Proposition 2, $(N :_R M)$ is a nonnil ideal of R. Hence, $(N :_R M)$ is a finitely-generated ideal of R, and hence, $N = (N :_R M)M$ is a finitely-generated submodule of M. Thus M is a nonnil-Noetherian R-module. □

Theorem 14. *Let R be a commutative ring, and M a finitely-generated faithful multiplication R-module with $M \in \mathbb{H}$. If each nonnil prime submodule of M has a power that is finitely-generated, then M is a nonnil-Noetherian R-module.*

Proof. Let P be a nonnil prime ideal of R. Then PM is a nonnil prime submodule of M by Proposition 2 and the fact that M is a cancellation module. Hence, there exists a positive integer t such that $(PM)^t = P^tM$ is a finitely-generated submodule of M. Hence, P^t is finitely-generated by [16, Lemma 3.5]. It follows from [14, Theorem 1.6] that R is a nonnil-Noetherian ring. Therefore, M is a nonnil-Noetherian R-module by Theorem 13. □

Proposition 15. *Let R be a commutative ring, and M a Noetherian multiplication R-module. If $P \subset Q$ are prime submodules of M such that there exists a prime submodule properly between P and Q, then there are infinitely many prime submodules of M properly between P and Q.*

Proof. Without loss of generality, we may assume that M is faithful, otherwise, by replacing R with $R/Ann(M)$, we can assume that M is faithful. If we set $p = (P :_R M)$, and $q = (Q :_R M)$, then $p \subset q$ are prime ideals of R by [22, Corollary 2.11]. Suppose that $N = IM$ is a prime submodule of M properly between P and Q. Then I is a prime ideal of R properly between p and q by [22, Corollary 2.11]. On the other hand, since M is a Noetherian R-module, it follows that R is a Noetherian ring. Hence, by [18, Theorem 144], there are infinitely many prime ideals of R properly between p and q. As there is a one-to-one correspondence between the prime ideals of R and the prime submodules of M, it follows that there are infinitely many prime submodules of M properly between P and Q. □

Theorem 16. *Let R be a commutative ring, and $M \in \mathbb{H}$ be a nonnil-Noetherian multiplication R-module. If $P \subset Q$ are prime submodules of M such that there exists a prime submodule properly between P and Q, then there are infinitely many prime submodules of M properly between P and Q.*

Proof. If we set $L = M/Nil(M)$, then L is a Noetherian R-module by Theorem 9. Suppose that $P \subset Q$ are prime submodules of M such that there exists a prime submodule N properly between P and Q. Then the prime submodule $N/Nil(M)$ is properly between the prime submodules $P/Nil(M) \subset Q/Nil(M)$ of the R-module L. Hence, there are infinitely many prime submodules of L properly between $P/Nil(M)$ and $Q/Nil(M)$ by Proposition 15. Therefore, there are infinitely many prime submodules of M properly between P and Q. □

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مدول‌های ناپوچ-نوتری روی حلقه‌های جابجایی

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در این مقاله رده‌ای از مدول‌ها که به رده مدول‌های نوتری نزدیک می‌باشد را معرفی می‌کنیم. فرض کنید R یک حلقه جابجایی و یکدار باشد و M یک R-مدول باشد بطوری که $Nil(M)$ یک زیرمدول اول تقسیم شده از M می‌باشد. M را یک مدول ناپوچ-نوتری می‌نامیم هرگاه هر زیرمدول غیر پوچ از M با تولید متناهی باشد. ثابت می‌کنیم که بسیاری از خواص مدول‌های نوتری برای مدول‌های ناپوچ-نوتری نیز برقرارند.

کلمات کلیدی: حلقه نوتری، مدول نوتری، زیرمدول با تولید متناهی، زیرمدول تقسیم‌شده، فی-مدول.