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A SHORT PROOF OF A RESULT OF NAGEL

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Abstract. Let $(R, \mathfrak{m})$ be a Gorenstein local ring, and M and N be two finitely-generated modules over R. Nagel proved that if M and N are in the same even liaison class, then one has $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(N)$ for all $i < \dim M = \dim N$. In this paper, we provide a short proof to this result.

1. Introduction

Let V_1 and V_2 be the subschemes of the projective scheme $\mathbb{P}^n$ over a field k, $I_X \subseteq I_{V_1} \cap I_{V_2}$ be an arithmetically Gorenstein ideal (for a closed subschemes Y of $\mathbb{P}^n$, and I_Y denote a saturated homogeneous ideal that defines Y scheme, theoretically). Then V_1 is (algebraically) linked to V_2 by X if and only if $[I_X : I_{V_1}] = I_{V_2}$ and $[I_X : I_{V_2}] = I_{V_1}$.

There is a more subtle relation between the evenly linked schemes, in the form of a necessary condition for the two schemes to be linked. Let V_1 and V_2 be two evenly-linked schemes of dimension r. This means that there are the schemes $W_0 = V_1, W_1, \ldots, W_s = V_2$ such that W_i and W_{i+1} are directly linked for all $i = 0, \ldots, s-1$, and s is even. Using reflexive sheaves and his notion of generalized divisors Hartshorne proved in 1991 that there is an integer p such that $H^i_{(x_0,\ldots,x_n)}(k[x_0, \ldots, x_n]/I_{V_1}) \cong H^i_{(x_0,\ldots,x_n)}(k[x_0, \ldots, x_n]/I_{V_2})(p)$ as graded modules for each $1 \leq i \leq r$ ([3, Proposition 4.5]). Here, for a module M over a Noetherian local ring $(R, \mathfrak{m})$, we use $H^i_{\mathfrak{m}}(M)$ to denote the i-th local cohomology module of M with respect to $\mathfrak{m}$.

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In 1998, Nagel generalized the above result to the case of linkage of ideals. Below, we review this notion. Let $\mathfrak{c} \subset R$ be an ideal, where R is a Gorenstein. Then $\mathfrak{c}$ is said to be a Gorenstein ideal if $\mathfrak{c}$ is a perfect ideal and $R/\mathfrak{c}$ is Gorenstein. The ideal $\mathfrak{c}$ is called a complete intersection of height c if it is generated by an R-regular sequence of length c. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \subset R$ be ideals such that $\mathfrak{c}$ is a Gorenstein ideal. Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be (directly) linked by $\mathfrak{c}$ if $\mathfrak{c} : \mathfrak{a} = \mathfrak{b}$ and $\mathfrak{c} : \mathfrak{b} = \mathfrak{a}$. Iterating the procedure, we say that an ideal $\mathfrak{b}$ is in the linkage class of an ideal $\mathfrak{a}$ if there are ideals $\mathfrak{c}_0 = \mathfrak{a}, \ldots, \mathfrak{c}_s = \mathfrak{b}$ such that $\mathfrak{c}_i$ and $\mathfrak{c}_{i+1}$ are directly linked for all $i = 0, \ldots, s-1$. If, in addition, s is even, we say that $\mathfrak{b}$ is in the even linkage class of $\mathfrak{a}$.

Let $\mathfrak{a}, \mathfrak{b} \subset R$ be two evenly-linked ideals, where $(R, \mathfrak{m})$ is a Gorenstein local ring. Then in 1998, Nagel generalized the Hartshorne's result, and proved that $H^i_{\mathfrak{m}}(R/\mathfrak{a}) \cong H^i_{\mathfrak{m}}(R/\mathfrak{b})$ for each $i < \dim R/\mathfrak{a} = \dim R/\mathfrak{b}$ ([4, Corollary 3.13]).

Finally, in 2005, after producing his notion of linkage of modules, Nagel obtained the module theoretical version of the mentioned result by means of resolutions of E-type and Q-type ([5, Corollary 6.1.]).

In what follows, we provide a short and simple proof to the Nagel's result.

2. Preliminary

The statement of the theorem mentioned below involves the notion of linkage of modules. For the convenience of the reader, we review this notion briefly below.

Throughout the paper, R denotes a local Gorenstein ring with maximal ideal $\mathfrak{m}$. Let M be a finitely-generated R-module. Then $K_M := \operatorname{Ext}_R^{\dim R - \dim M}(M, R)$ is said to be the canonical module of M. The module M is said to be a quasi-Gorenstein R-module if it is a perfect R-module such that $M \cong K_M$.

Let C be a quasi-Gorenstein module of codimension c. We denote by $\operatorname{Epi}(C)$ the set of R-module homomorphisms $\varphi : C \longrightarrow M$, where M is an R-module and $\operatorname{im} \varphi$ has the same dimension as C. Given a homomorphisms $\varphi \in \operatorname{Epi}(C)$, we want to construct a new homomorphism $L_C(\varphi)$. Ultimately, we will see that this construction gives the map $\operatorname{Epi}(C) \longrightarrow \operatorname{Epi}(C) \cup \{0\}$. In order to do so, note that the exact sequence $0 \to \ker \varphi \to C \to \operatorname{im} \varphi \to 0$ induces the exact sequence:

$$\operatorname{Ext}^c_R(\operatorname{im} \varphi, R) \longrightarrow \operatorname{Ext}^c_R(C, R) \xrightarrow{\psi} \operatorname{Ext}^c_R(\ker \varphi, R).$$

By assumption, there is an isomorphism $\alpha : C \longrightarrow \operatorname{Ext}^c_R(C, R)$. Thus we obtain the homomorphism $\psi := \psi' \circ \alpha : C \longrightarrow \operatorname{Ext}^c_R(\ker \varphi, R)$, which we denote by $L_C(\varphi)$. Then a map $L_C : \mathrm{Epi}(C) \longrightarrow \mathrm{Epi}(C) \cup \{0\}$, $\varphi \mapsto L_C(\varphi)$ is called the linking map with respect to C (and α). Here, 0 denotes the trivial homomorphism $C \longrightarrow 0_R$. Nagel showed, for each non-injective homomorphism $\varphi \in \mathrm{Epi}(C)$, that there is an exact sequence,

$$0 \longrightarrow K_{\mathrm{im}\,\varphi} \longrightarrow C \longrightarrow \mathrm{im}\, L_C(\varphi) \longrightarrow 0. \tag{2.1}$$

We say the two R-modules M and N are module-linked in one step by the quasi-Gorenstein module C if there are homomorphisms $\varphi, \psi \in \mathrm{Epi}(C)$ such that:

(1) $M = \mathrm{im}\,\varphi$, $N = \mathrm{im}\,\psi$ and
(2) $M \cong \mathrm{im}\, L_C(\psi)$, $N \cong \mathrm{im}\, L_C(\varphi)$.

Module liaison or simply liaison is the equivalence relation generated by the direct module linkage. Its equivalence classes are called the (module) liaison classes. Thus the two modules M and M', belong to the same liaison class if there are modules $N_0 = M, N_1, \ldots, N_{s-1}, N_s = M'$ such that N_i and N_{i+1} are directly linked for all $i = 0, \ldots, s-1$. In this case, we say that M and M' are linked in s steps. If s is even, then M and M' are said to be evenly-linked.

Using the exact sequence 2.1, Nagel proved the following lemma.

Lemma 2.1. *If the modules M and N are directly linked by the quasi-Gorenstein module C, then there is an exact sequence of R-modules*

$$0 \longrightarrow K_M \longrightarrow C \longrightarrow N \longrightarrow 0. \tag{2.2}$$

Using the so-called resolutions of E-type and Q-type, finally, Nagel obtained the following result.

Theorem 2.2. *Let M and N be two finitely-generated R-modules. If M and N are in the same even liaison class, then for all $i < \dim M = \dim N$, one has:*

$$H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(N).$$

Note that, by the definition of two linked modules, we have $\dim M = \dim C = \dim N$ whenever M and N are linked by a quasi-Gorenstein module C.

3. Simple proof of Theorem 2.2

Theorem 2.2 admits a very short proof, which is based on the Lemma [6, Lemma 4 .2] of Schenzel in the case of linkage of ideals. The following lemma generalizes Schenzel's lemma to the linkage of modules. The proof is mutatis mutandis the same as that for [6, Lemma 4 .2]. However, for the reader's convenience, we reprove it in the case of linkage of modules.

In the sequel, we use E to denote the injective hull of $R/\mathfrak{m}$.

Lemma 3.1. *Let M and N be two directly-linked modules by the quasi-Gorenstein module C. Set $n := \dim M = \dim N$. Then there is an exact sequence of R-modules,*

$$0 \longrightarrow H^{n-1}_{\mathfrak{m}}(N) \longrightarrow H^n_{\mathfrak{m}}(K_M) \longrightarrow \operatorname{Hom}_R(M, E) \longrightarrow 0,$$

and isomorphisms $H^{i-1}_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(K_M)$ for all $i < n$.

Proof. Applying the functor $H^0_{\mathfrak{m}}(-)$ to the exact sequence 2.2 in conjunction with the perfectness of C, one deduces that $H^{i-1}_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(K_M)$ for all $i < n$. To prove the remaining part of the lemma, first note that by applying the functor $\operatorname{Hom}_R(-, R)$ to the exact sequence 2.2 together with [2, Corollary 3.5.11], one obtains the exact sequence:

$$0 \longrightarrow K_N \longrightarrow C \longrightarrow \operatorname{Ext}^c_R(K_M, R) \longrightarrow \operatorname{Ext}^{c+1}_R(N, R) \longrightarrow 0,$$

where $c := \dim R - \dim C$. Now, using this and the exact sequence $0 \longrightarrow K_N \longrightarrow C \longrightarrow M \longrightarrow 0$, one gets the exact sequence

$$0 \longrightarrow M \longrightarrow \operatorname{Ext}^c_R(K_M, R) \longrightarrow \operatorname{Ext}^{c+1}_R(N, R) \longrightarrow 0.$$

This completes the proof in view of the Local Duality Theorem [1, 11.2.5]. □

Now we are ready to give a short and simple proof of Theorem 2.2.

Proof of Theorem 2.2. By assumption, there exists an R-module A such that M is linked to A and A is linked to N. Hence, using the above lemma, we have the isomorphisms $H^i_{\mathfrak{m}}(M) \cong H^{i+1}_{\mathfrak{m}}(K_A)$ and $H^i_{\mathfrak{m}}(N) \cong H^{i+1}_{\mathfrak{m}}(K_A)$ for all $i < n - 1$, where $n = \dim M = \dim N$. This shows that $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(N)$ for all $i < n - 1$. To obtain the assertion in case where $i = n - 1$, one notices that we have the following two exact sequences:

$$0 \longrightarrow H^{n-1}_{\mathfrak{m}}(M) \longrightarrow H^n_{\mathfrak{m}}(K_A) \longrightarrow \operatorname{Hom}_R(A, E) \longrightarrow 0,$$
$$0 \longrightarrow H^{n-1}_{\mathfrak{m}}(N) \longrightarrow H^n_{\mathfrak{m}}(K_A) \longrightarrow \operatorname{Hom}_R(A, E) \longrightarrow 0.$$

□

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اثباتی کوتاه به نتیجه‌ای از ناگل

نعمت‌اله شیرمحمدی
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فرض کنید $(R, \mathfrak{m})$ یک حلقه موضعی گرنشتاین و M, N مدول‌های متناهی مولد روی R باشند. ناگل ثابت کرده است که اگر M و N در کلاس پیوند زوج یکسان باشند، آنگاه، به ازای هر $i < \dim M = \dim N$، $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(N)$. در این مقاله، اثباتی کوتاه برای این نتیجه داده می‌شود.

کلمات کلیدی: پیوند، کوهمولوژی موضعی، مدول شبه-گرنشتاین