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FINITE GROUPS WITH FIVE NON-CENTRAL CONJUGACY CLASSES

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ABSTRACT. Let G be a finite group and $Z(G)$ be the center of G. For a subset A of G, we define $k_G(A)$, the number of conjugacy classes of G that intersect A non-trivially. In this paper, we verify the structure of all finite groups G which satisfy the property $k_G(G - Z(G)) = 5$, and classify them.

1. Introduction

Let M be a normal subgroup of a finite group G. The influence of the arithmetic structure of conjugacy classes of G, like conjugacy class sizes, the number of conjugacy classes or the number of conjugacy class sizes, on the structure of G is an extensively studied question in group theory. Shi [5], Shahryari and Shahabi [4] and Riese and Shahabi [3] determined the structure of M, when M is the union of 2, 3 or 4 conjugacy classes of G, respectively. Qian et al. [2] considered the opposite extreme situation that contains almost all conjugacy classes of G, and determined the structure of the whole group, when there are at most 3 conjugacy classes outside M. In particular, You et al. [7] classified all finite groups G, when there are at most 4 conjugacy classes of G outside the center of G. In this paper, we continue the work in [7] and verify the structure of finite groups G, when there are 5 conjugacy classes outside the center of G. Let $Z(G)$ be the center of G. For an element x of G, we will let $o(x)$ to denote the order of x and x^G

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to denote the conjugacy class of x in G. For $A \subseteq G$, let $k_G(A)$ be the number of conjugacy classes of G that intersect A non-trivially. Recall that $K \rtimes H$ is a semidirect product of K and H with normal subgroup K. In particular, the Frobenius group with kernel K and complement H is denoted by $K \times_f H$. Also, the semi-dihedral group of order 2^n is denoted by $SD_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, bab = a^{2^{n-2}-1} \rangle$. All further unexplained notations are standard. The purpose of this paper is to classify all finite groups in which there are five non-central conjugacy classes.

Theorem 1.1. *Let G be a non-abelian finite group. Then $k_G(G - Z(G)) = 5$ if and only if G is isomorphic to one of the following groups:*

(1) $PSL(2, 7)$;
(2) $SL(2, 3)$;
(3) D_{16}, Q_{16} *or* SD_{16};
(4) D_{18};
(5) $(\mathbb{Z}_3)^2 \times_f \mathbb{Z}_2$;
(6) $(\mathbb{Z}_3)^2 \times_f \mathbb{Z}_4$;
(7) $(\mathbb{Z}_3)^2 \times_f Q_8$;
(8) $\mathbb{Z}_4 \rtimes \mathbb{Z}_3$.

2. Preliminaries

In this section, we present some preliminary results that will be used in the proof of the Theorem 1.1.

Lemma 2.1. (See [1]) *Let G be a finite group and $k_G(G)$ be the number of conjugacy classes of G. Then*

(1) *If $k_G(G) = 1$, then $G \cong \{1\}$;*
(2) *If $k_G(G) = 2$, then $G \cong \mathbb{Z}_2$;*
(3) *If $k_G(G) = 3$, then G is isomorphic to $\mathbb{Z}_3$ or S_3;*
(4) *If $k_G(G) = 4$, then G is isomorphic to $\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, D_{10}$ or A_4;*
(5) *If $k_G(G) = 5$, then G is isomorphic to $\mathbb{Z}_5, D_8, Q_8, D_{14}, S_4, A_5, \mathbb{Z}_7 \times_f \mathbb{Z}_3$ or $\mathbb{Z}_5 \times_f \mathbb{Z}_4$;*
(6) *If $k_G(G) = 6$, then G is isomorphic to $\mathbb{Z}_6, D_{12}, \mathbb{Z}_4 \rtimes \mathbb{Z}_3, D_{18}, (\mathbb{Z}_3)^2 \times_f \mathbb{Z}_2, (\mathbb{Z}_3)^2 \times_f \mathbb{Z}_4, (\mathbb{Z}_3)^2 \times_f Q_8$ or $PSL(2, 7)$.*

Lemma 2.2. [2, Lemma 1.3] *If G possesses an element x with $|C_G(x)| = 4$, then a Sylow 2-subgroup P of G is the dihedral, semi-dihedral or generalized quaternion group. In particular, $|P/P'| = 4$ and P has a cyclic subgroup of order $|P|/2$.*

Proposition 2.3. [2, Proposition 2.1] *If N is a normal subgroup of a finite non-abelian group G, then $k_G(G-N)=1$ if and only if G is a Frobenius group with kernel N and $|N|=|G|/2$.*

Lemma 2.4. [7, Lemma 6], *Let G be a finite group and K, N be two normal subgroups of G with $|K/N|=p$, where p is prime. If $|C_G(x)|=p$ for any $x\in K-N$, then K is a Frobenius group with kernel N.*

3. The proof of Theorem 1.1.

To prove our main result, Theorem 1.1, we first state the following theorem.

Theorem 3.1. *There is no finite non-abelian group G such that $G/Z(G)$ is abelian and $k_G(G-Z(G))=5$.*

Proof. Let $G-Z(G)=x^G\cup y^G\cup z^G\cup w^G\cup t^G$. Since $k_{G/Z(G)}(G/Z(G)-Z(G)/Z(G))\leq k_G(G-Z(G))=5$, we have $k_{G/Z(G)}(G/Z(G))\leq 6$. It follows from Lemma 2.1 that $G/Z(G)$ is an elementary abelian 2-group of order 4. Hence, $k_{G/Z(G)}(G/Z(G)-Z(G)/Z(G))=3$. We may assume that $xZ(G)=x^G$, $yZ(G)=y^G$, $zZ(G)=wZ(G)=tZ(G)=z^G\cup w^G\cup t^G$. It then implies that $|C_G(x)|=|C_G(y)|=4$, $|C_G(z)|=|C_G(w)|=|C_G(t)|=12$ or $|C_G(x)|=|C_G(y)|=4$, $|C_G(z)|=|C_G(w)|=16$, $|C_G(t)|=8$ or $|C_G(x)|=|C_G(y)|=4$, $|C_G(z)|=|C_G(w)|=24$, $|C_G(t)|=6$. In the first two cases, we have $|Z(G)|=2$ or 4 and hence $|G|=8$ or 16. It follows from Lemma 2.2 that $|Z(G)|=2$. Therefore, $|G|=8$ and G is isomorphic to D_8 or Q_8, which forces $k_G(G-Z(G))=3$, a contradiction. In the third case, we have $|Z(G)|=2$, and get the same contradiction. □

By Theorem 3.1, since there is no group G with abelian central factor and $k_G(G-Z(G))=5$, we are ready to prove our main result, Theorem 1.1, and note that $G/Z(G)$ is not abelian.

Let $G-Z(G)=x^G\cup y^G\cup z^G\cup w^G\cup t^G$. We consider the following two cases:

Case 1. If G is non-solvable, then $G/Z(G)$ is non-solvable too. Since $k_{G/Z(G)}(G/Z(G)-Z(G)/Z(G))\leq k_G(G-Z(G))=5$, we conclude that $k_{G/Z(G)}(G/Z(G))\leq 6$. It follows from Lemma 2.1 that $G/Z(G)$ is isomorphic to $PSL(2,7)$ or A_5. If $G/Z(G)\cong PSL(2,7)$, then $k_G(G-Z(G))=k_{G/Z(G)}(G/Z(G)-Z(G)/Z(G))=5$, and we have $k_G(xZ(G))=k_G(yZ(G))=k_G(zZ(G))=k_G(wZ(G))=k_G(tZ(G))=1$. Therefore, $|C_G(x)|=|C_{G/Z(G)}(xZ(G))|$, $|C_G(y)|=|C_{G/Z(G)}(yZ(G))|$, $|C_G(z)|=|C_{G/Z(G)}(zZ(G))|$, $|C_G(w)|=|C_{G/Z(G)}(wZ(G))|$ and $|C_G(t)|=|C_{G/Z(G)}(tZ(G))|$. We may assume that $o(xZ(G))=3$, $o(yZ(G))=$

7, $o(zZ(G)) = 7$, $o(wZ(G)) = 2$ and $o(tZ(G)) = 4$. Then we have $|C_{G/Z(G)}(xZ(G))| = 3$, $|C_{G/Z(G)}(yZ(G))| = 7$, $|C_{G/Z(G)}(zZ(G))| = 7$, $|C_{G/Z(G)}(wZ(G))| = 8$ and $|C_{G/Z(G)}(tZ(G))| = 4$. Hence, $|C_G(x)| = 3$, $|C_G(y)| = |C_G(z)| = 7$, $|C_G(w)| = 8$ and $|C_G(t)| = 4$. Therefore, $|Z(G)| = 1$ and $G \cong PSL(2,7)$.

Now, let $G/Z(G) \cong A_5$. Then, $k_{G/Z(G)}(G/Z(G)) = 5$. Since $k_G(G - Z(G)) = 5$, $G/Z(G)$ has three non-trivial conjugacy classes as the same as three conjugacy classes of $G-Z(G)$. Moreover, $G/Z(G)$ has one non-trivial conjugacy class, that is the union of two remaining conjugacy classes of $G-Z(G)$. Since the order of the centralizer of representative of each of three conjugacy classes of $G - Z(G)$ in G is 3, 4 or 5, we conclude that $|Z(G)| = 1$. Therefore, $G \cong A_5$, a contradiction.

Case 2. If G is solvable, then we have $k_{G/Z(G)}(G/Z(G)) \leq 6$. It then implies that $G/Z(G)$ is isomorphic to one of the following groups: $S_3, D_{10}, A_4, Q_8, D_8, D_{14}, S_4, \mathbb{Z}_7 \times_f \mathbb{Z}_3, \mathbb{Z}_5 \times_f \mathbb{Z}_4, D_{12}, D_{18}, (\mathbb{Z}_3)^2 \times_f \mathbb{Z}_2$, $(\mathbb{Z}_3)^2 \times_f \mathbb{Z}_4, (\mathbb{Z}_3)^2 \times_f Q_8$ or $\mathbb{Z}_4 \rtimes \mathbb{Z}_3$. Hence, we consider the following subcases:

Subcase 2.1. Suppose that $G/Z(G) \cong S_3$ and $K/Z(G)$ be a Sylow 3-subgroup of $G/Z(G)$. Then, $K \lhd G$, $k_{G/Z(G)}(G/Z(G)) = 3$ and $|G/K| = 2$.

If $k_G(G - K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K and $|G/K| = 2$. Therefore $|Z(G)| = 1$ and hence $G \cong S_3$, a contradiction.

If $k_G(G - K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K-Z(G) = z^G \cup w^G \cup t^G$. It then implies that $|x^G|+|y^G| = |G-K| = |G|/2$ and $|z^G| + |w^G| + |t^G| = |K - Z(G)| = |G|/3$. Thus, we have either $|C_G(x)| = |C_G(y)| = 4$, $|C_G(z)| = |C_G(w)| = |C_G(t)| = 9$ or $|C_G(x)| = |C_G(y)| = 4$, $|C_G(z)| = 6$, $|C_G(w)| = |C_G(t)| = 12$. In the first case, we have $|Z(G)| = 1$ and so $G \cong S_3$, a contradiction. In the second case, we have $|Z(G)| = 2$. Therefore, $|G| = 12$, and by [6], we get a contradiction.

If $k_G(G - K) = 3$ or $k_G(G - K) = 4$, then by a similar argument, we get a contradiction.

Subcase 2.2. Suppose that $G/Z(G) \cong D_{10}$ and $K/Z(G)$ be a Sylow 5-subgroup of $G/Z(G)$. Then, $K \lhd G$, $k_{G/Z(G)}(G/Z(G)) = 4$ and $|G/K| = 2$.

If $k_G(G - K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K and $|G/K| = 2$. Therefore $|Z(G)| = 1$ and $G \cong D_{10}$, a contradiction.

If $k_G(G - K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K-Z(G) = z^G \cup w^G \cup t^G$. So we have $|x^G|+|y^G| = |G-K| = |G|/2$ and $|z^G|+|w^G|+|t^G| = |K-Z(G)| = 2|G|/5$. Hence $|C_G(x)| = |C_G(y)| = 4$.

Since $|K/Z(G)| = 5$, we have $|C_G(z)| = 5$ and $|C_G(w)| = |C_G(t)| = 10$. Therefore, $|Z(G)| = 1$ and $G \cong D_{10}$, a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G-K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $|x^G| + |y^G| + |Z^G| = |G-K| = |G|/2$ and $|w^G| + |t^G| = |K - Z(G)| = 2|G|/5$. Therefore, $|C_G(w)| = |C_G(t)| = 5$ and by Lemma 2.4, K is a Frobenius group with kernel $Z(G)$, a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Therefore, we have $|C_G(t)| = 5/2$, a contradiction.

Subcase 2.3. Suppose that $G/Z(G) \cong A_4$. Then, $k_{G/Z(G)}(G/Z(G)) = 4$. Let $K/Z(G)$ be a Sylow 2-subgroup of $G/Z(G)$. We conclude that $K \lhd G$, $|G/K| = 3$ and $k_{G/Z(G)}(G/Z(G) - K/Z(G)) = 2$. Hence, $k_G(G - K) \geq 2$.

If $k_G(G - K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $|x^G| + |y^G| = |G - K| = 2|G|/3$ and $|z^G| + |w^G| + |t^G| = |K - Z(G)| = |G|/4$. It then implies that $|C_G(x)| = |C_G(y)| = 3$ and by Lemma 2.4, G is a Frobenius group with kernel K. Therefore, $|Z(G)| = 1$ and $G \cong A_4$, a contradiction.

If $k_G(G - K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So we have $|x^G| + |y^G| + |Z^G| = |G - K| = 2|G|/3$ and $|w^G| + |t^G| = |K - Z(G)| = |G|/4$. Hence we have either $|C_G(x)| = |C_G(y)| = 6$, $|C_G(z)| = 3$, $|C_G(w)| = |C_G(t)| = 8$ or $|C_G(x)| = |C_G(y)| = 6$, $|C_G(z)| = 3$, $|C_G(w)| = 6$, $|C_G(t)| = 12$. In the first case, we have $|Z(G)| = 1$ and so $G \cong A_4$, a contradiction. In the second case, we have $|Z(G)| = 3$. Therefore, $|G| = 36$ and by [6], we get a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Hence, we have $|C_G(x)| = |C_G(y)| = |C_G(z)| = |C_G(w)| = 6$, $|C_G(t)| = 4$ or $|C_G(x)| = |C_G(y)| = |C_G(z)| = 9$, $|C_G(w)| = 3$, $|C_G(t)| = 4$ or $|C_G(x)| = |C_G(y)| = 12$, $|C_G(z)| = 6$, $|C_G(w)| = 3$, $|C_G(t)| = 4$. In the first case, we have $|Z(G)| = 2$ and so $G \cong SL(2, 3)$. For the other two cases, $|Z(G)| = 1$, a contradiction.

Subcase 2.4. Suppose that $G/Z(G) \cong Q_8$. In this case $k_{G/Z(G)}(G/Z(G)) = 5$. If $|Z(G)| = 1$, then $G \cong Q_8$, which forces $k_G(G - Z(G)) = 3$, a contradiction. Now suppose that $|Z(G)| > 1$. Let $K/Z(G)$ be a cyclic subgroup of $G/Z(G)$ of order 4. Then, we have $K \lhd G$, $k_{G/Z(G)}(G/Z(G) - K/Z(G)) = 2$ and $k_{G/Z(G)}(K/Z(G) - Z(G)/Z(G)) = 3$. It follows that $k_G(G - K) = 2$ and $k_G(K - Z(G)) = 3$. We may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. It then implies that either $|C_G(x)| = |C_G(y)| = 4$, $|C_G(z)| = |C_G(w)| = |C_G(t)| = 8$ or $|C_G(x)| = |C_G(y)| = 4$, $|C_G(z)| = |C_G(w)| = 16$, $|C_G(t)| = 4$. In the both cases, we have $|Z(G)| = 2$ or 4 and so $|G| = 16$ or 32. Now, Lemma 2.2 implies that $|Z(G)| = 2$ and $|G| = 16$. Therefore, G is isomorphic to D_{16}, Q_{16} or SD_{16} .

Subcase 2.5. Suppose that $G/Z(G) \cong D_8$. Using the same argument as in Subcase 2.4, we conclude that G is isomorphic to D_{16}, Q_{16} or SD_{16}.

Subcase 2.6. Suppose that $G/Z(G) \cong D_{14}$ and $K/Z(G)$ be a Sylow 7-subgroup of $G/Z(G)$. Then, $K \lhd G$, $k_{G/Z(G)}(G/Z(G)) = 5$ and $|G/K| = 2$.

If $k_G(G - K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K and $|G/K| = 2$. Hence, $|Z(G)| = 1$ and $G \cong D_{14}$, a contradiction.

If $k_G(G - K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $|x^G| + |y^G| = |G - K| = |G|/2$ and $|z^G| + |w^G| + |t^G| = |K - Z(G)| = 3|G|/7$. Therefore, $|C_G(z)| = |C_G(w)| = |C_G(t)| = 7$ and by Lemma 2.4, we have K is a Frobenius group with kernel $Z(G)$, a contradiction.

If $k_G(G - K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{3}{7}$. Suppose that $|C_G(w)| = 7a$ and $|C_G(t)| = 7b$, for some integers a and b. Then, $\frac{1}{a} + \frac{1}{b} = 3$, which has no solution, a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Therefore, we have $|C_G(t)| = \frac{7}{3}$, a contradiction.

Subcase 2.7. Suppose that $G/Z(G) \cong S_4$. Then, $k_{G/Z(G)}(G/Z(G)) = 5$. Since $k_G(G - Z(G)) = 5$, $G/Z(G)$ has three non-trivial conjugacy classes as the same as three conjugacy classes of $G - Z(G)$. Also $G/Z(G)$ has one non-trivial conjugacy class that is the union of two remaining conjugacy classes of $G - Z(G)$. Since the order of the centralizer of representative of each of four non-trivial conjugacy classes of $G/Z(G)$ is 3, 4 or 8, we have the following two cases:

(1) The order of the centralizer of representative of one of five non-central conjugacy classes of G is 3. In this case, using a similar argument mentioned before, we conclude that $|Z(G)| = 1$. Therefore, $G \cong S_4$, a contradiction.

(2) The order of the centralizer of representative of none of five non-central conjugacy classes of G is 3. So, G has three non-central conjugacy classes, in which the orders of the centralizers of representatives of them are 4, 4 and 8. Thus, we have $\frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{a} + \frac{1}{b} + \frac{1}{24} = 1$, where a and b are the orders of the centralizers of representatives of two other conjugacy classes. This equality holds if $a = b = 6$ or $a = 4$, $b = 12$. In the first case, we get $|Z(G)| = 2$. Therefore, $|G| = 48$ and by [6], we have a contradiction. In the second case, $|Z(G)| = 2$ or 4.

Therefore, $|G| = 48$ or 96 and by [6], we have a contradiction.

Subcase 2.8. Assume that $G/Z(G) \cong \mathbb{Z}_7 \times_f \mathbb{Z}_3$. In this case $k_{G/Z(G)}(G/Z(G)) = 5$. If $|Z(G)| = 1$, then $G \cong \mathbb{Z}_7 \times_f \mathbb{Z}_3$, a contradiction. Now, suppose that $|Z(G)| > 1$. Let $K/Z(G)$ be a Sylow 7-subgroup of $G/Z(G)$. Then we have $K \lhd G$ and $|G/K| = 3$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G-K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{2}{3}$. Let $|C_G(x)| = 3a$ and $|C_G(y)| = 3b$, for some integers a and b. Then $\frac{1}{a} + \frac{1}{b} = 2$ and so $|C_G(x)| = |C_G(y)| = 3$. Now, by Lemma 2.4, G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G-K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{2}{7}$. Let $|C_G(w)| = 7a$ and $|C_G(t)| = 7b$, for some integers a and b. Then $\frac{1}{a} + \frac{1}{b} = 2$ and so $|C_G(w)| = |C_G(t)| = 7$. Now, by Lemma 2.4, K is a Frobenius group with kernel $Z(G)$, a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Then, we have $|C_G(t)| = 7/2$, a contradiction.

Subcase 2.9. Suppose that $G/Z(G) \cong \mathbb{Z}_5 \times_f \mathbb{Z}_4$. In this case $k_{G/Z(G)}(G/Z(G)) = 5$. If $|Z(G)| = 1$, then $G \cong \mathbb{Z}_5 \times_f \mathbb{Z}_4$, a contradiction. Now, suppose that $|Z(G)| > 1$ and $K/Z(G)$ be a Sylow 5-subgroup of $G/Z(G)$. Then, we have $K \lhd G$ and $|G/K| = 4$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G-K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. It then implies that $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{3}{4}$ and $\frac{1}{|C_G(z)|} + \frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{1}{5}$. Thus, either $|C_G(x)| = 2$, $|C_G(y)| = 4$, $|C_G(z)| = |C_G(w)| = |C_G(t)| = 15$ or $|C_G(x)| = 2$, $|C_G(y)| = 4$, $|C_G(z)| = |C_G(w)| = 20$, $|C_G(t)| = 10$. In the first case, we have $|Z(G)| = 1$, a contradiction. In the second case, $|Z(G)| = 2$. Therefore, $|G| = 40$ and by [6], we get a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G-K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} = \frac{3}{4}$ and $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{1}{5}$. Then we conclude that either $|C_G(x)| = |C_G(y)| = |C_G(z)| = 4$, $|C_G(w)| = |C_G(t)| = 10$ or $|C_G(x)| = 2$, $|C_G(y)| = |C_G(z)| = 8$, $|C_G(w)| = |C_G(t)| = 10$. In the both cases, $|Z(G)| = 2$ and so $|G| = 40$, which is not possible.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Therefore, $|C_G(t)| = 5$ and by Lemma 2.4, K is a Frobenius group with kernel $Z(G)$, a contradiction.

Subcase 2.10. Suppose that $G/Z(G) \cong D_{12}$. In this case $k_{G/Z(G)}(G/Z(G)) = 6$. If $|Z(G)| = 1$, then $G \cong D_{12}$, a contradiction. Now, suppose that $|Z(G)| > 1$ and $K/Z(G)$ be a Sylow 3-subgroup of $G/Z(G)$. Then, we have $K \lhd G$ and $|G/K| = 4$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{3}{4}$ and $\frac{1}{|C_G(t)|} + \frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{1}{6}$. Then, we conclude that either $|C_G(x)| = 2$, $|C_G(y)| = 4$, $|C_G(z)| = |C_G(w)| = |C_G(t)| = 18$ or $|C_G(x)| = 2$, $|C_G(y)| = 4$, $|C_G(z)| = |C_G(w)| = 24$, $|C_G(t)| = 12$. In the both cases, $|Z(G)| = 2$ and $|G| = 24$, which is not possible.

If $k_G(G-K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} = \frac{3}{4}$ and $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{1}{6}$. It then implies that either $|C_G(x)| = |C_G(y)| = |C_G(z)| = 4$, $|C_G(w)| = |C_G(t)| = 12$ or $|C_G(x)| = |C_G(y)| = 8$, $|C_G(z)| = 2$, $|C_G(w)| = |C_G(t)| = 12$. In the first case, we have $|Z(G)| = 2$ or 4 and so $|G| = 24$ or 48, which is not possible. In the second case, $|Z(G)| = 2$ and hence $|G| = 24$, a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Thus, we have $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$, $|C_G(w)| = 4$, $|C_G(t)| = 6$ or $|C_G(x)| = |C_G(y)| = |C_G(z)| = 12$, $|C_G(w)| = 2$, $|C_G(t)| = 6$ or $|C_G(x)| = |C_G(y)| = 8$, $|C_G(z)| = |C_G(w)| = 4$, $|C_G(t)| = 6$. In each case, $|Z(G)| = 2$ and so $|G| = 24$, which is not possible.

Subcase 2.11. Suppose that $G/Z(G) \cong D_{18}$. In this case $k_{G/Z(G)}(G/Z(G)) = 6$. If $|Z(G)| = 1$, then $G \cong D_{18}$. Now, suppose that $|Z(G)| > 1$ and $K/Z(G)$ be a Sylow 3-subgroup of $G/Z(G)$. Then, we have $K \lhd G$ and $|G/K| = 2$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{1}{2}$ and $\frac{1}{|C_G(z)|} + \frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{4}{9}$. It then implies that $|C_G(x)| = |C_G(y)| = 4$, $|C_G(z)| = 9$ and $|C_G(w)| = |C_G(t)| = 6$. Therefore, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} = \frac{1}{2}$ and $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{4}{9}$. Hence, we have either $|C_G(x)| = |C_G(y)| =$

$|C_G(z)| = 6$, $|C_G(w)| = 9$, $|C_G(t)| = 3$ or $|C_G(x)| = |C_G(y)| = 8$, $|C_G(z)| = 4$, $|C_G(w)| = 9$, $|C_G(t)| = 3$. In the first case, we have $|Z(G)| = 3$. Therefore, $|G| = 54$ and by [6], we get a contradiction. In the second case, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Therefore, $|C_G(t)| = \frac{9}{4}$, a contradiction.

Subcase 2.12. Suppose that $G/Z(G) \cong (\mathbb{Z}_3)^2 \times_f \mathbb{Z}_2$. In this case $k_{G/Z(G)}(G/Z(G)) = 6$. If $|Z(G)| = 1$, then $G \cong (\mathbb{Z}_3)^2 \times_f \mathbb{Z}_2$. Now, suppose that $|Z(G)| > 1$ and $K/Z(G)$ be a Sylow 3-subgroup of $G/Z(G)$. Then, we have $K \lhd G$ and $|G/K| = 2$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{1}{2}$ and $\frac{1}{|C_G(z)|} + \frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{4}{9}$. Thus $|C_G(x)| = |C_G(y)| = 4$, $|C_G(z)| = 9$, $|C_G(w)| = |C_G(t)| = 6$. Therefore $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} = \frac{1}{2}$ and $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{4}{9}$. It then implies that either $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$, $|C_G(w)| = 9$, $|C_G(t)| = 3$ or $|C_G(x)| = |C_G(y)| = 8$, $|C_G(z)| = 4$, $|C_G(w)| = 9$, $|C_G(t)| = 3$. In the first case, we have $|Z(G)| = 3$ and so $|G| = 54$, which is not possible. In the second case, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Therefore, $|C_G(t)| = \frac{9}{4}$, a contradiction.

Subcase 2.13. Suppose that $G/Z(G) \cong (\mathbb{Z}_3)^2 \times_f \mathbb{Z}_4$. In this case $k_{G/Z(G)}(G/Z(G)) = 6$. If $|Z(G)| = 1$, then $G \cong (\mathbb{Z}_3)^2 \times_f \mathbb{Z}_4$. Now, suppose that $|Z(G)| > 1$ and $K/Z(G)$ be a Sylow 3-subgroup of $G/Z(G)$. Then, we have $K \lhd G$ and $|G/K| = 4$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{3}{4}$ and $\frac{1}{|C_G(z)|} + \frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{2}{9}$. Thus, we conclude that $|C_G(x)| = 2$, $|C_G(y)| = 4$, $|C_G(z)| = 9$, $|C_G(w)| = |C_G(t)| = 18$. Therefore, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} = \frac{3}{4}$ and $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{2}{9}$. It then implies that either $|C_G(x)| = |C_G(y)| = |C_G(z)| = 4$, $|C_G(w)| = |C_G(t)| = 9$ or $|C_G(x)| = 2$,

$|C_G(y)| = |C_G(z)| = 8$, $|C_G(w)| = |C_G(t)| = 9$. In the both cases, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Therefore, $|C_G(t)| = \frac{9}{2}$, a contradiction.

Subcase 2.14. Suppose that $G/Z(G) \cong (\mathbb{Z}_3)^2 \times_f Q_8$. In this case $k_{G/Z(G)}(G/Z(G)) = 6$. If $|Z(G)| = 1$, then $G \cong (\mathbb{Z}_3)^2 \times_f Q_8$. Now suppose that $|Z(G)| > 1$ and $K/Z(G)$ be a Sylow 3-subgroup of $G/Z(G)$. Then, we have $K \lhd G$ and $|G/K| = 8$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{7}{8}$, which has no integer solution, a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} = \frac{7}{8}$ and $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{1}{9}$. It then implies that $|C_G(x)| = 8$, $|C_G(y)| = 2$, $|C_G(z)| = 4$, $|C_G(w)| = |C_G(t)| = 18$. Therefore, $|Z(G)| = 2$ and $|G| = 144$. Hence, by [6], we get a contradiction.

If $k_G(G-K) = 4$, then we may assume that $G - K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. Then, we conclude that either $|C_G(x)| = |C_G(y)| = |C_G(z)| = 8$, $|C_G(w)| = 2$, $|C_G(t)| = 9$ or $|C_G(x)| = |C_G(y)| = |C_G(z)| = 4$, $|C_G(w)| = 8$, $|C_G(t)| = 9$. In the both cases, $|Z(G)| = 1$, a contradiction.

Subcase 2.15. Suppose that $G/Z(G) \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_3$. In this case $k_{G/Z(G)}(G/Z(G)) = 6$. If $|Z(G)| = 1$, then $G \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_3$. Now, suppose that $|Z(G)| > 1$ and $K/Z(G)$ be a Sylow 2-subgroup of $G/Z(G)$. Then we have $K \lhd G$ and $|G/K| = 3$.

If $k_G(G-K) = 1$, then it follows from Proposition 2.3 that G is a Frobenius group with kernel K. Hence, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 2$, then we may assume that $G - K = x^G \cup y^G$ and $K - Z(G) = z^G \cup w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} = \frac{2}{3}$ and $\frac{1}{|C_G(z)|} + \frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{1}{4}$. Thus, we conclude that either $|C_G(x)| = |C_G(y)| = 3$, $|C_G(z)| = |C_G(w)| = |C_G(t)| = 12$ or $|C_G(x)| = |C_G(y)| = 3$, $|C_G(z)| = 8$, $|C_G(w)| = |C_G(t)| = 16$. In the first case, we have $|Z(G)| = 3$ and so $|G| = 36$, which is not possible. In the second case, $|Z(G)| = 1$, a contradiction.

If $k_G(G-K) = 3$, then we may assume that $G - K = x^G \cup y^G \cup z^G$ and $K - Z(G) = w^G \cup t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} = \frac{2}{3}$ and $\frac{1}{|C_G(w)|} + \frac{1}{|C_G(t)|} = \frac{1}{4}$. It then implies that either $|C_G(x)| = |C_G(y)| = 6$, $|C_G(z)| = 3$, $|C_G(w)| = |C_G(t)| = 8$ or $|C_G(x)| = |C_G(y)| = 6$,

$|C_G(z)| = 3$, $|C_G(w)| = 6$, $|C_G(t)| = 12$. In the first case, we have $|Z(G)| = 1$, a contradiction. In the second case, $|Z(G)| = 3$ and hence $|G| = 36$, which is not possible.

If $k_G(G-K) = 4$, then we may assume that $G-K = x^G \cup y^G \cup z^G \cup w^G$ and $K - Z(G) = t^G$. So, we have $\frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|} + \frac{1}{|C_G(z)|} + \frac{1}{|C_G(w)|} = \frac{2}{3}$ and $\frac{1}{|C_G(t)|} = \frac{1}{4}$. Thus, $|C_G(x)| = |C_G(y)| = |C_G(z)| = |C_G(w)| = 6$, $|C_G(t)| = 4$ or $|C_G(x)| = 3$, $|C_G(y)| = |C_G(z)| = |C_G(w)| = 9$, $|C_G(t)| = 4$ or $|C_G(x)| = 3$, $|C_G(y)| = 6$, $|C_G(z)| = |C_G(w)| = 12$, $|C_G(t)| = 4$. In the first case, we have $|Z(G)| = 2$ and so $|G| = 24$. Therefore, $G \cong SL(2,3)$. In the other two cases, we have $|Z(G)| = 1$, a contradiction.

Now the proof of Theorem 1.1 is complete.

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FINITE GROUPS WITH FIVE NON-CENTRAL CONJUGACY CLASSES

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گروه‌های متناهی با پنج کلاس تزویج نامرکزی

مهدی رضائی و زینب فروزان فر
مرکز آموزش عالی فنی و مهندسی بوئین زهرا

فرض کنید G یک گروه متناهی باشد و $Z(G)$ مرکز G باشد. برای یک زیرمجموعه A از G، $K_G(A)$ را برابر تعداد کلاس‌های تزویج G که اشتراکشان با A غیربدیهی است، تعریف می‌کنیم. در این مقاله، ما ساختار تمامی گروه‌های متناهی G که در خاصیت $K_G(G-Z(G))=۵$ صدق می‌کنند را بررسی کرده و آنها را رده‌بندی می‌کنیم.

کلمات کلیدی: گروه متناهی، گروه فروبنیوس، کلاس تزویج.