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FUZZY OBSTINATE IDEALS IN MV-ALGEBRAS

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Abstract. In this paper, we introduce the notion of fuzzy obstinate ideals in MV-algebras. Some properties of fuzzy obstinate ideals are given. Not only we give some characterizations of fuzzy obstinate ideals, but also bring the extension theorem of fuzzy obstinate ideal of an MV-algebra A. We investigate the relationships between fuzzy obstinate ideals and the other fuzzy ideals of an MV-algebra. We describe the transfer principle for fuzzy obstinate ideals in terms of level subsets. In addition, we show that if μ is a fuzzy obstinate ideal of A such that $\mu(0) \in [0, 1/2]$, then A/μ is a Boolean algebra. Finally, we define the notion of a normal fuzzy obstinate ideal and investigate some of its properties.

1. Introduction

MV-algebras were introduced by C. C. Chang (1958). MV-algebras are algebraic counterparts of the Łukasiewicz infinite many valued propositional logic [1].

The concept of fuzzy set was introduced by Zadeh [15]. This idea has been applied to other algebraic structures such as semigroups, groups, rings, ideals, modules, vector spaces and topologies.

C. S. Hoo obtained various results of fuzzy ideals of BCI, BCK and MV-algebras, and introduced fuzzy implicative and Boolean ideals of MV-algebras [6], [7].

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In [4], results regarding obstinate ideals of MV-algebras were obtained. Also, fuzzy equivalence relations were defined and studied by Murali in 1989 [13].

In this paper, we introduce the notion of fuzzy obstinate ideals in MV-algebras and study several properties of fuzzy obstinate ideals. We give some characterizations of fuzzy obstinate ideals and establish the extension theorem of this class of ideals. In addition, we investigate the relationships between obstinate fuzzy ideals and the other fuzzy ideals of an MV-algebra, and use a level set of fuzzy set in an MV-algebra. We show that if μ is a fuzzy obstinate ideal of A such that $\mu(0) \in [0, 1/2]$, then A/μ is a Boolean algebra. Finally, we introduce normal obstinate ideals and investigate some properties.

We recollect some definitions and results which will be used in the sequel.

Definition 1.1. [1, Definition 1.1] An MV-algebra is a structure $(A, \oplus, {}^*, 0)$, where $\oplus$ is a binary operation, *, is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:

$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* \oplus a = 0^*$,
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that 0^* is usually denoted as 1 and the auxiliary operations $\odot$, $\wedge$ and $\vee$ are defined as

$$x \odot y = (x^* \oplus y^*)^*, \quad x \vee y = x \oplus (x^* \odot y),$$

$$x \wedge y = x \odot (x^* \oplus y).$$

It is shown that $(A, \odot, 1)$ is an abelian monoid and the structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. We recall that for any two elements $x, y \in A$, $x \leq y$ if and only if $x^* \oplus y = 1$ [1, Definition 1.1].

Definition 1.2. [1, Definition 4.1] An ideal of an MV-algebra A is a non-empty subset I of A satisfying the following conditions:

$(I1)$ If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$,
$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.

We denote by $Id(A)$, the set of ideals of an MV-algebra A.

Definition 1.3. [2] A proper ideal P is a prime ideal of an MV-algebra A, if $x \wedge y \in P$, then $x \in P$ or $y \in P$, for all $x, y \in A$. We denote by $Spec(A)$, the set of all prime ideals of A.

Lemma 1.4. [14, Theorem 2.3] *In each MV-algebra, the following properties hold for all $x, y, z \in A$:*

(1) $x \leq y$ *if and only if* $y^* \leq x^*$,
(2) *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$, $x \odot z \leq y \odot z$,
(3) $x \leq y$ *if and only if* $x^* \oplus y = 1$ *if and if* $x \odot y^* = 0$,
(4) $x, y \leq x \oplus y$, $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$, $x^n = x \odot x \odot \cdots \odot x \leq x$, $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,
(5) $x \odot y^* \wedge y \odot x^* = 0$,
(6) $(x \oplus y) \wedge (x \oplus z) = x \oplus (y \wedge z)$.

Definition 1.5. [2] Let X and Y be MV-algebras. A function $f : X \to Y$ is called homomorphism of MV-algebras if and only if

(1) $f(0) = 0$,
(2) $f(x \oplus y) = f(x) \oplus f(y)$,
(3) $f(x^*) = (f(x))^*$.

Definition 1.6. [4, Definition 2.1] A proper ideal of A is called an obstinate ideal A, if $x, y \notin I$ implies $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in A$.

Definition 1.7. [15] A fuzzy set in A is a mapping $\mu : A \to [0, 1]$. Let μ be a fuzzy set in A. For $t \in [0, 1]$, the set $\mu^t = \{x \in A : \mu(x) \geq t\}$ is called a level subset of μ.

For any fuzzy sets μ, ν in A, the binary relation $\subseteq$ is defined as

$$\mu \subseteq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

Definition 1.8. [15] Let X, Y be two sets, μ be a fuzzy subset of X, μ' be a fuzzy subset of Y and $f : X \to Y$ be a homomorphism. The image of μ under f, denoted by $f(\mu)$, is a fuzzy set of Y, defined by: for all $y \in Y$, $f(\mu)(y) = sup_{x \in f^{-1}(y)} \mu(x)$, if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(y) = 0$ if $f^{-1}(y) = \emptyset$.

The preimage of μ' under f, denoted by $f^{-1}(\mu')$, is a fuzzy set of X defined by: for all $x \in X$, $f^{-1}(\mu')(x) = \mu'(f(x))$.

Definition 1.9. [6] Let A be an MV-algebra. Then, a fuzzy set μ in A is a fuzzy ideal of A, if it satisfies

(MV1) $\mu(0) \geq \mu(x)$, for all $x \in A$,
(MV2) $\mu(y) \geq \mu(x) \wedge \mu(y \odot x^*)$, for all $x, y \in A$.

Lemma 1.10. [6, Proposition 2.1] *Let A be an MV-algebra and $\mu : A \to [0, 1]$ be a fuzzy set on A. Then, μ is called a fuzzy ideal on A, if and only if*

(1) $\mu(x) \leq \mu(0)$, *for all* $x \in A$,
(2) $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$, *for all* $x, y \in A$,

(3) *If* $x \leq y$, *then* $\mu(x) \geq \mu(y)$.

Theorem 1.11. [3, Proposition 3.3] *Let* μ *be a fuzzy ideal in* A. *For any* $x, y, z \in A$, *the following hold:*

(1) $\mu(x \oplus y) = \mu(x) \wedge \mu(y)$,
(2) $\mu(x \odot y) \geq \mu(x) \wedge \mu(y)$,
(3) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,
(4) $\mu(nx) = \mu(x)$, *for any* $n \in \mathbb{N}$,
(5) $\mu(x \vee y) = \mu(x) \wedge \mu(y)$.

Definition 1.12. Let μ be a fuzzy ideal in A.

- [6, Definition 2] μ is called a fuzzy prime ideal, if $\mu(x \wedge y) = \mu(x) \vee \mu(y)$, for all $x, y \in A$.
- [7, Definition 2.4] μ is called a fuzzy Boolean ideal, if $\mu(x \wedge x^*) = \mu(0)$, for all $x \in A$.

Theorem 1.13. [6, Theorem 3.8] *Suppose that* μ *is both fuzzy Boolean and fuzzy prime ideal. Then, the following holds for all* $x, y, z \in A$:

(i) $\mu(x \vee y) = \mu(x) \wedge \mu(y)$,
(ii) $\mu(x \wedge y) = \mu(x) \vee \mu(y)$,
(iii) $\mu(x) \wedge \mu(x^*) = \mu(1)$,
(iv) $\mu(x) \vee \mu(x^*) = \mu(0)$,
(v) $\mu(x \odot y^*) \geq \mu(x \odot z^*) \wedge \mu(z \odot y^*)$,
(vi) *If* $\mu(x \odot y^*) = \mu(0)$, *then* $\mu(x) \geq \mu(y)$ [5, Lemma 2.11].

Theorem 1.14. [5, Definition 3.1] *Let* A *be an* MV*-algebra. A fuzzy relation* θ *from* $A \times A$ *to* $[0, 1]$, *is called a fuzzy congruence in* A *if it satisfies the following conditions:*

(C1) $\theta(0, 0) = \theta(x, x)$,
(C2) $\theta(x, y) = \theta(y, x)$,
(C3) $\theta(x, z) \geq \theta(x, y) \wedge \theta(y, z)$,
(C4) $\theta(x \oplus z, y \oplus z) \geq \theta(x, y)$,
(C5) $\theta(x^*, y^*) = \theta(x, y)$, *for all* $x, y, z \in A$. *Let* μ *be a fuzzy ideal in* A *and* $x \in A$. *In the following, let* μ^x *denote the fuzzy congruence class of* x *by* θ_μ *in* A *and* A/μ *the fuzzy quotient set by* $\theta_{\mu(x,y)}$, *where* $\theta_\mu(x, y) = \mu(x \odot y^*) \wedge \mu(y \odot x^*)$ [5, Lemma 3.9].

Corollary 1.15. [5, Corollary 3.12] *If* μ *is a fuzzy ideal, then* $\mu^x = \mu^y$ *if and only if* $x \sim_{\mu_{\mu(0)}} y$, *where*

$$x \sim_{\mu_{\mu(0)}} y \text{ if and only if } x \odot y^* \in \mu_{\mu(0)} \text{ and } y \odot x^* \in \mu_{\mu(0)}.$$

Let μ *be a fuzzy ideal in* A. *For any* $\mu^x, \mu^y \in A/\mu$, *we define* $\mu^x \oplus \mu^y = \mu^{x \oplus y}$, $(\mu^x)^* = \mu^{x^*}$. *Hence,* $A/\mu = (A/\mu, \oplus, ^*, \mu^0, \mu^1)$ *is an* MV*-algebra.*

Theorem 1.16. [5, Theorem 3.16] *Let μ be a fuzzy ideal in A. Define a mapping $f : A \to A/\mu$ by $f(x) = \mu^x$. Then,*

(1) *f is a surjective homomorphism,*
(2) *$Ker(f) = \mu^{\mu(0)}$,*
(3) *A/μ is isomorphic to the MV-algebra $A/\mu^{\mu(0)}$.*

Theorem 1.17. [12, Theorem 1] *For every fuzzy subset μ of X, μ is a fuzzy $\boldsymbol{P}$-set if and only if for all $t \in [0, 1]$, $\mu^t \neq \emptyset$, then μ^t is a $\boldsymbol{P}$-set, where a subset X has a property $\boldsymbol{P}$.*

2. Fuzzy obstinate ideals in MV-algebras

In this section, we introduce fuzzy obstinate ideals in MV-algebras and give a characterization of fuzzy obstinate ideals. We also prove extension theorem of fuzzy obstinate ideals in MV-algebras.

From now onwards, $(A, \oplus, {}^*, 0)$ or simply A is an MV-algebra.

Definition 2.1. A non-constant fuzzy ideal μ is called a fuzzy obstinate ideal of A if and only if

$$min\{\mu(x \odot y^*), \mu(y \odot x^*)\} \geq min\{1 - \mu(x), 1 - \mu(y)\},$$

for all $x, y \in A$.

The following proposition gives a characterization of fuzzy obstinate ideals.

Proposition 2.2. *A non-constant fuzzy ideal μ of an MV-algebra A is a fuzzy obstinate ideal if and only if it satisfies the following condition:*

$$\mu(x^*) \geq 1 - \mu(x), \text{ for all } x \in A.$$

Proof. Suppose that μ is a fuzzy obstinate ideal of A. Since $x \leq 1$, by fuzzy ideal properties, we obtain $1 - \mu(x) \leq 1 - \mu(1)$, and we conclude that

$$\begin{aligned} & \mu(x^*) = \mu(x^* \odot 1) \\ \geq \;& min\{\mu(x^* \odot 1), \mu(x \odot 1^*)\} \\ \geq \;& min\{1 - \mu(x), 1 - \mu(1)\} \\ = \;& 1 - \mu(x). \end{aligned}$$

Conversely, let $\mu(x^*) \geq 1 - \mu(x)$, for all $x \in A$. By Lemma 1.4 (4), $x^* \odot y \leq x^*$ and $y^* \odot x \leq y^*$. Then by Lemma 1.10 (3), we have

$$min\{1 - \mu(x), 1 - \mu(y)\} \leq 1 - \mu(x) \leq \mu(x^*) \leq \mu(x^* \odot y)$$

and $min\{1 - \mu(x), 1 - \mu(y)\} \leq 1 - \mu(y) \leq \mu(y^*) \leq \mu(y^* \odot x)$. Thus, $min\{1 - \mu(x), 1 - \mu(y)\} \leq min\{\mu(x^* \odot y), \mu(y^* \odot x)\}$. Hence, μ is a fuzzy obstinate ideal of A. □

The following example shows that fuzzy obstinate ideals in MV-algebras exist, while a fuzzy ideal may not be a fuzzy obstinate ideal A.

Example 2.3. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and * as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, ^*, 0, 1)$ is an MV-algebra [11, Example 1].

(i) Define a fuzzy set μ in A by $\mu(0) = 0.8$, $\mu(a) = 0.7$, and $\mu(1) = \mu(b) = 0.5$. Obviously, μ is a fuzzy obstinate ideal of A.

(ii) Define a fuzzy set μ' in A by $\mu'(0) = 0.8$, $\mu'(a) = 0.5$, and $\mu'(1) = \mu'(b) = 0.3$.

Obviously, μ' is a fuzzy ideal in A. Since $\mu'(b^*) = \mu'(a) = 0.5 < 1 - \mu'(b) = 0.7$, hence μ' is not obstinate ideal of A.

Lemma 2.4. *(Extension theorem of fuzzy obstinate ideals) Suppose that μ and ν are two non-constant fuzzy ideals such that $\mu \subseteq \nu$. If μ is a fuzzy obstinate ideal, then ν is also a fuzzy obstinate ideal.*

Proof. Let μ be a fuzzy obstinate ideal such that $\mu \subseteq \nu$. We show that ν is a fuzzy obstinate ideal. Since μ is a fuzzy obstinate ideal, $\mu(x^*) \geq 1 - \mu(x)$, for all $x \in A$. Since $\mu \subseteq \nu$, so $\mu(x) \leq \nu(x)$, for all $x \in A$. It then follows that

$$\nu(x^*) \geq \mu(x^*) \geq 1 - \mu(x) \geq 1 - \nu(x).$$

Hence, $\nu(x^*) \geq 1 - \nu(x)$, for all $x \in A$. Thus, ν is a fuzzy obstinate ideal. □

Remark 2.5. (i) If μ_i, $(i \in \Gamma)$ is fuzzy obstinate ideal, then $\wedge_{i \in \Gamma} \mu_i$ is not fuzzy obstinate ideal, in general.

Consider fuzzy obstinate ideal μ in Example 2.3 (i), and define a fuzzy set μ' by $\mu'(0) = 0.8$, $\mu'(b) = 0.7$ and $\mu'(1) = \mu'(a) = 0.3$. We can easily show that μ' is a fuzzy obstinate ideal of A. We have $\mu \wedge \mu'(0) = 0.8$, $\mu \wedge \mu'(b) = 0.5$ and $\mu \wedge \mu'(1) = \mu \wedge \mu'(a) = 0.3$. Since

$0.5 = \mu \wedge \mu'(a^*) < 1 - \mu \wedge \mu'(a) = 1 - 0.3 = 0.7$, so $\mu \wedge \mu'$ is not fuzzy obstinate ideal of A.

(ii) If $\wedge_{i\in\Gamma}\mu_i$ is a fuzzy obstinate ideal of A, since $\wedge_{i\in\Gamma}\mu_i \leq \mu_i$, then by Lemma 2.4, we get that μ_i is fuzzy obstinate ideal of A, for all $i \in \Gamma$.

(iii) Let μ_i be fuzzy obstinate ideal of A, for all $i \in \Gamma$. Then, by Lemma 2.4, $\vee_{i\in\Gamma}\mu_i$ is fuzzy obstinate ideal.

Theorem 2.6. *Let $f : X \to Y$ be onto MV-homomorphism. Then the preimage of a fuzzy obstinate ideal μ under f is also a fuzzy obstinate ideal of A.*

Proof. Suppose that μ is a fuzzy obstinate ideal of Y. Then, for all $x, y \in X$, we have

$$\begin{aligned} & min\{f^{-1}(\mu)(x \odot y^*), f^{-1}(\mu)(y \odot x^*)\} \\ = & \; min\{\mu(f(x) \odot f(y)^*), \mu(f(y) \odot f(x)^*)\} \\ \geq & \; min\{1 - \mu(f(x)), 1 - \mu(f(y))\} \\ = & \; min\{1 - f^{-1}(\mu)(x), 1 - f^{-1}(\mu)(y)\}. \end{aligned}$$

Thus $f^{-1}(\mu)$ is a fuzzy obstinate ideal of X. □

Proposition 2.7. *Let $f : X \to Y$ be an onto MV-homomorphism. Then the image $f(\mu)$ of a fuzzy obstinate ideal μ is again a fuzzy obstinate ideal.*

Proof. It is sufficient to show that for all $y_1, y_2 \in Y$, we have

$$min\{f(\mu)(y_1 \odot y_2^*), f(\mu)(y_2 \odot y_1^*)\} \geq min\{1 - f(\mu)(y_1), 1 - f(\mu)(y_2)\}.$$

Let $y_1, y_2 \in Y$. Assume that $min\{1 - f(\mu)(y_1), 1 - f(\mu)(y_2)\} = t$. Then $sup_{a\in f^{-1}(y_1)}\mu(a) = f(\mu)(y_1) \leq 1 - t$ and so $1 - \mu(a) \geq t$, for all $a \in f^{-1}(y_1)$. Similarly, $1 - \mu(b) \geq t$, for all $b \in F^{-1}(y_2)$. Now, $a \in f^{-1}(y_1)$ and $b \in f^{-1}(y_2)$ imply that $f(a \odot b^*) = y_1 \odot y_2^*$, whence $f(\mu)(y_1 \odot y_2^*) \geq \mu(a \odot b^*)$ and $f(\mu)(y_2 \odot y_1^*) \geq \mu(b \odot a^*)$. Thus,

$$\begin{aligned} & min\{f(\mu)(y_1 \odot y_2^*), f(\mu)(y_2 \odot y_1^*)\} \\ \geq & \; min\{\mu(a \odot b^*), \mu(b \odot a^*)\} \\ \geq & \; min\{1 - \mu(a), 1 - \mu(b)\} \\ \geq & \; t. \end{aligned}$$

Therefore $f(\mu)$ is a fuzzy obstinate ideal. □

Theorem 2.8. *A non-empty subset I of X is an obstinate ideal if and only if the characteristic function χ_I is a fuzzy obstinate ideal of A.*

Proof. Assume that I is an obstinate ideal of A. We will prove that χ_I is a fuzzy obstinate ideal. For all $x, y \in A$, we show that

$$min\{\chi_I(x \odot y^*), \chi_I(y \odot x^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\}.$$

If $x \in I$ or $y \in I$, then we have $min\{1 - \chi_I(x), 1 - \chi_I(y)\} = 0$ and

$$min\{\chi_I(x \odot y^*), \chi_I(y \odot x^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\}.$$

If $x \notin I$ and $y \notin I$, then $min\{1 - \chi_I(x), 1 - \chi_I(y)\} = 1$, since I is an obstinate ideal, we obtain $x \odot y^* \in I$ and $y \odot x^* \in I$. So $min\{\chi_I(x \odot y^*), \chi_I(y \odot x^*)\} = 1$. We conclude that

$$min\{\chi_I(x \odot y^*), \chi_I(y \odot x^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\}.$$

Assume that χ_I is a fuzzy obstinate ideal. We prove that I is an obstinate ideal. Letting $x, y \notin I$, we have $\chi_I(x) = 0 = \chi_I(y)$. Since χ_I is a fuzzy obstinate ideal, we have

$$min\{\chi_I(x \odot y^*), \chi_I(y \odot x^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\} = 1.$$

Therefore weobtain $\chi_I(x \odot y^*) = \chi_I(y \odot x^*) = 1$. Hence, $x \odot y^* \in I$ and $y \odot x^* \in I$. □

3. The relations between fuzzy obstinate ideals and the other fuzzy ideals in MV-algebras

Now, we describe the transfer principle [12] for fuzzy obstinate ideals in terms of level subsets.

Theorem 3.1. (i) *If fuzzy subset μ of MV-algebra A is a fuzzy obstinate ideal, of A, then $\mu^t = \{x \in A \ : \ \mu(x) \geq t\}$ is either empty or an obstinate ideal for every $t \in [0, \frac{1}{2}]$.*

(ii) *If every nonempty level subset μ^t with $t \in [0, 1]$ is an obstinate ideal, then μ is a fuzzy obstinate ideal of A.*

Proof. (i) Assume that μ is a fuzzy obstinate ideal of A. By [10, Theorem 1], we have μ is non-constant if and only if μ^t is proper. Now, let $t \in [0, \frac{1}{2}]$ and $x \in \mu^t$. Then, $\mu(x) \geq t$. Since μ is a fuzzy ideal, $\mu(0) \geq \mu(x)$. Therefore $0 \in \mu^t$. Let $x, y \notin \mu^t$. We show that $x \odot y^* \in \mu^t$ and $y \odot x^* \in \mu^t$. Since $x, y \notin \mu^t$, $\mu(x) < t$ and $\mu(y) < t$. Since μ is a fuzzy obstinate ideal, we have

$$\begin{aligned} \mu(x \odot y^*) &\geq \min\{\mu(x \odot y^*), \mu(y \odot x^*)\} \\ &\geq \min\{1 - \mu(x), 1 - \mu(y)\} \\ &> 1 - t \\ &\geq t, \end{aligned}$$

for every $t \leq \frac{1}{2}$.

Similarly we get $\mu(y \odot x^*) \geq t$. Hence, $x \odot y^* \in \mu^t$ and $y \odot x^* \in \mu^t$, for every $t \leq \frac{1}{2}$. Thus μ^t is an obstinate ideal of A, for every $t \in [0, \frac{1}{2}]$.
(ii) Assume that for every $t \in [0,1]$, μ^t is an obstinate ideal of A. We will prove that μ is a fuzzy obstinate ideal. It is easy to prove that for all $x \in A$, $\mu(0) \geq \mu(x)$, since $0 \leq x \in \mu^{\mu(x)}$ and μ^t is an ideal of A, so $0 \in \mu^{\mu(x)}$. Hence, $\mu(0) \geq \mu(x)$, for $x \in A$. Now, we show that $\min\{\mu(x \odot y^*), \mu(y \odot x^*)\} \geq \min\{1-\mu(x), 1-\mu(y)\}$. If not, then there exist $a, b \in A$ such that $\min\{\mu(a \odot b^*), \mu(b \odot a^*)\} < \min\{1-\mu(a), 1-\mu(b)\}$. Setting $t_0 = \frac{1}{2}(\min\{\mu(a \odot b^*), \mu(b \odot a^*)\} + \min\{1-\mu(a), 1-\mu(b)\}$, we have $\min\{\mu(a \odot b^*), \mu(b \odot a^*))\} < t_0 < \min\{1-\mu(a), 1-\mu(b)\}$. Then, we conclude that $\mu(a \odot b^*) < t_0$ or $\mu(b \odot a^*) < t_0$. Also, $t_0 < 1-\mu(a)$ and $t_0 < 1-\mu(b)$.

We consider the following two cases:

Case 1. If $t_0 > \frac{1}{2}$, then we conclude that $\mu(a) < 1-t_0 < t_0$ and $\mu(b) < 1-t_0 < t_0$. Also, since $\mu(a \odot b^*) < t_0$ or $\mu(b \odot a^*) < t_0$, hence $a \odot b^* \notin \mu^{t_0}$ or $b \odot a^* \notin \mu^{t_0}$, for $a \notin \mu^{t_0}$ and $b \notin \mu^{t_0}$, which is a contradiction.

Case 2. If $t_0 \leq \frac{1}{2}$, then $\mu(a \odot b^*) < t_0 \leq \frac{1}{2} \leq 1-t_0$ or $\mu(b \odot a^*) < t_0 \leq \frac{1}{2} \leq 1-t_0$. Also, $\mu(a) < 1-t_0$ and $\mu(b) < 1-t_0$. Hence, $a \odot b \notin \mu^{1-t_0}$ or $b \odot a \notin \mu^{1-t_0}$, for $a \notin \mu^{1-t_0}$ and $b \notin \mu^{1-t_0}$, which is a contradiction. Therefore, μ is a fuzzy obstinate ideal and the proof is complete. □

Corollary 3.2. *Let μ be a fuzzy ideal of an* MV*-algebra A. The level ideal $\mu^{\mu(0)} = \{x \in A : \mu(x) = \mu(0)\}$ is an obstinate ideal of A if μ is a fuzzy obstinate ideal of A with $\mu(0) \in [0, \frac{1}{2}]$.*

Theorem 3.3. *If μ is a fuzzy obstinate ideal of A such that $\mu(0) \in [0, \frac{1}{2}]$, then A/μ is a Boolean* MV*-algebra.*

Proof. If μ is a fuzzy obstinate ideal of A, then by Corollary 3.2, we conclude that $\mu^{\mu(0)}$ is an obstinate ideal of A. It then follows that $A/\mu^{\mu(0)}$ is a Boolean MV-algebra (see [4, Corollary 2.2]). Using Theorem 1.16, we conclude that A/μ is a Boolean algebra. □

Theorem 3.4. *Let μ be a fuzzy obstinate ideal of A. Then μ is a fuzzy Boolean ideal of A.*

Proof. It is sufficient to show that $\mu(x \wedge x^*) = \mu(0)$. Since $0 \leq x \wedge x^*$, then by fuzzy ideal property, $\mu(0) \geq \mu(x \wedge x^*)$.
Since μ is a fuzzy obstinate ideal of A and by Proposition 2.2 and

Theorem 1.11 (1), (5), we have

$$\begin{array}{rcl} \mu(x \wedge x^*) & = & \mu((x^* \vee x)^*) \\ & \geq & 1 - \mu(x^* \vee x) \\ & = & 1 - \mu(x^*) \wedge \mu(x) \\ & = & 1 - \mu(x^* \oplus x) \\ & = & 1 - \mu(1) \\ & \geq & 1 - \mu(0) \\ & \geq & \mu(0). \end{array}$$

Hence, $\mu(x \wedge x^*) = \mu(0)$. Thus, μ is a fuzzy Boolean ideal of A. □

The following example shows that the converse of Theorem 3.4 is not true in general.

Example 3.5. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then, $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [11, Example 4]. Consider $\mu(1) = \mu(0) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = 0.4$. Obviously, μ is a fuzzy Boolean ideal. Since $\mu(a^*) = \mu(d) = 0.4 < 1 - \mu(a) = 1 - 0.4 = 0.6$, hence μ is not an obstinate ideal of A.

Theorem 3.6. *If μ is a fuzzy prime ideal and fuzzy Boolean ideal of A and $\mu(1) \geq \frac{1}{2}$, then μ is a fuzzy obstinate ideal of A.*

Proof. Let μ be a fuzzy prime and fuzzy Boolean ideal of A. By Theorem 1.13 (iii), we conclude that

$$\begin{array}{rcl} \mu(x^*) & \geq & \mu(x) \wedge \mu(x^*) \\ & = & \mu(1) \\ & \geq & 1 - \mu(1) \\ & \geq & 1 - \mu(x). \end{array}$$

Thus μ is a fuzzy obstinate ideal of A. □

Lemma 3.7. *Let μ be a non-constant fuzzy ideal of A. Then, the following are equivalent:*

(a) *μ is a fuzzy prime ideal of A,*

(b) *for all* $x, y \in A$, *if* $\mu(x \wedge y) = \mu(0)$, *then* $\mu(x) = \mu(0)$ *or* $\mu(y) = \mu(0)$,
(c) *for all* $x, y \in A$, $\mu(x \odot y^*) = \mu(0)$ *or* $\mu(y \odot x^*) = \mu(0)$.

Proof. (a) $\Rightarrow$ (b) Let μ be a fuzzy prime ideal of A. Let $x, y \in A$ such that $\mu(x \wedge y) = \mu(0)$. Then, $\mu(x) \vee \mu(y) = \mu(x \wedge y) = \mu(0)$ and hence $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$.

(b) $\Rightarrow$ (c) Using Lemma 1.4 (5), we obtain for all $x, y \in A$ that, $x \odot y^* \wedge y \odot x^* = 0$. Then, $\mu(x \odot y^* \wedge y \odot x^*) = \mu(0)$ and by (b), we conclude that $\mu(x \odot y^*) = \mu(0)$ or $\mu(y \odot x^*) = \mu(0)$.

(c) $\Rightarrow$ (a) Let $\mu(x \odot y^*) = \mu(0)$. By Lemma 1.4 (4), (6), we have

$$y \leq (x \vee y) \wedge (y \oplus (x^* \odot y)) = (x \oplus (x^* \odot y)) \wedge (y \oplus (x^* \odot y)) = (x \wedge y) \oplus (x^* \odot y).$$

Since μ is a fuzzy ideal, it then follows from Theorem 1.11 (1) that

$$\begin{aligned} \mu(y) &\geq \mu[(x \wedge y) \oplus (x^* \odot y)] \\ &= \mu(x \wedge y) \wedge \mu(x^* \odot y) \\ &= \mu(x \wedge y) \wedge \mu(0) \\ &= \mu(x \wedge y). \end{aligned}$$

Since $x \wedge y \leq y$, $\mu(y) \leq \mu(x \wedge y)$. Thus, we obtain $\mu(y) = \mu(x \wedge y)$. From Definition 1.9, we know that $\mu(y) \geq \mu(x) \wedge \mu(x^* \odot y) = \mu(x)$. Finally, we have $\mu(x) \vee \mu(y) = \mu(x \wedge y)$. Therefore, μ is a fuzzy prime ideal of A. □

Theorem 3.8. *If* μ *is fuzzy prime ideal such that* $\mu(x) \geq \frac{1}{2}$, *for all* $x \in A$, *then* μ *is a fuzzy obstinate ideal of* A.

Proof. Let μ be a fuzzy prime ideal. It then follows from Lemma 3.7 that $\mu(x \odot y^*) = \mu(0)$ or $\mu(y \odot x^*) = \mu(0)$. We consider the following two cases:
Case 1. If $\mu(x \odot y^*) = \mu(0)$, since $y \odot x^* \leq y$, $\mu(y \odot x^*) \geq \mu(y)$, then

$$\begin{aligned} \mu(y \odot x^*) &= \mu(0) \wedge \mu(y \odot x^*) \\ &= \mu(x \odot y^*) \wedge \mu(y \odot x^*) \\ &\geq \mu(y) \\ &\geq 1 - \mu(y) \\ &\geq (1 - \mu(x)) \wedge (1 - \mu(y)). \end{aligned}$$

Case 2. If $\mu(y \odot x^*) = \mu(0)$, since $x \odot y^* \leq x$, $\mu(x \odot y^*) \geq \mu(x)$, then

$$\begin{aligned}
\mu(x \odot y^*) &= \mu(0) \wedge \mu(x \odot y^*) \\
&= \mu(x \odot y^*) \wedge \mu(y \odot x^*) \\
&\geq \mu(x) \\
&\geq 1 - \mu(x) \\
&\geq (1 - \mu(x)) \wedge (1 - \mu(y)).
\end{aligned}$$

Thus, $\mu(x \odot y^*) \wedge \mu(y \odot x^*) \geq 1 - \mu(x) \wedge 1 - \mu(y)$. Therefore, μ is a fuzzy obstinate ideal of A. □

The following example shows that the converse of Theorem 3.8 is not true in general.

Example 3.9. In Example 2.3 (i), it is clear that μ is a fuzzy obstinate ideal. Since $0.8 = \mu(0) = \mu(a \wedge b) \neq \mu(a) \vee \mu(b) = 0.7 \vee 0.5 = 0.7$, hence μ is not a fuzzy prime ideal of A.

The following example shows that a fuzzy prime ideal with $\mathrm{Im}\mu \subseteq [0, \frac{1}{2})$ may not be a fuzzy obstinate ideal of A.

Example 3.10. Consider MV-algebra A in Example 2.3. Define a fuzzy set μ in A by $\mu(0) = \mu(a) = 0.4$, $\mu(1) = \mu(b) = 0.2$. It can be easily shown that μ is a fuzzy prime ideal but is not a fuzzy obstinate ideal, because $0.2 = \mu(b) = \mu(a^*) < 1 - \mu(a) = 1 - 0.4 = 0.6$.

4. Normal fuzzy obstinate ideals of MV-algebras

Definition 4.1. A fuzzy obstinate ideal μ of A is said to be normal if there exists an element $x_0 \in A$ such that $\mu(x_0) = 1$.

Clearly, a fuzzy obstinate ideal μ is normal if and only if $\mu(0) = 1$. Also, any fuzzy obstinate ideal containing a normal fuzzy obstinate ideal, is normal.

Example 4.2. In Example 2.3, we define a fuzzy set μ in A by $\mu(0) = 1$ and $\mu(1) = \mu(a) = \mu(b) = 0.5$. It is easy to verify that μ is a normal fuzzy obstinate ideal of A.

Theorem 4.3. *Let μ be a fuzzy obstinate ideal of A such that $\mu(1) < 1$. Then the fuzzy set $\bar{\mu}$, where $\bar{\mu}(x) = \mu(x) + 1 - \mu(0)$ for all $x \in A$, is a normal fuzzy obstinate ideal of A containing μ.*

Proof. Obviously, $\bar{\mu}(x) \in [0, 1]$ for every $x \in A$ and $\bar{\mu}(0) = 1$ and $\bar{\mu}(0) \geq \mu(x) + 1 - \mu(0) = \bar{\mu}(x)$. In [9, Lemma 3.3], it is shown that $\bar{\mu}$

is fuzzy ideal of A. It is sufficient to prove that $\bar{\mu}$ is a fuzzy obstinate ideal of A. Let $x, y \in A$. We have

$$\begin{aligned}
\bar{\mu}(x \odot y^*) \wedge \bar{\mu}(y \odot x^*) &= [\mu(x \odot y^*) + 1 - \mu(0)] \\
&\wedge [\mu(y \odot x^*) + 1 - \mu(0)] \\
&= (1 - \mu(0)) + [\mu(x \odot y^*) \wedge \mu(y \odot x^*)] \\
&\geq 1 - \mu(0) + [1 - \mu(x)) \wedge 1 - \mu(y)] \\
&= 1 - \mu(0) + (1 - \mu(x))] \\
&\wedge [1 - \mu(0) + (1 - \mu(y))] \\
&\geq [1 - (\mu(x) + 1 - \mu(0))] \\
&\wedge [1 - (\mu(y) + 1 - \mu(0))] \\
&= (1 - \bar{\mu}(x)) \wedge (1 - \bar{\mu}(y)).
\end{aligned}$$

Thus, $\bar{\mu}$ is a fuzzy obstinate ideal of A. Clearly, $\mu \subseteq \bar{\mu}$, which completes the proof. □

Corollary 4.4. $(\bar{\bar{\mu}}) = \bar{\mu}$ *for any fuzzy obstinate ideal* μ *of* A. *If* μ *is normal, then* $\bar{\mu} = \mu$.

We denote the set of all normal fuzzy obstinate ideals of A by $N(A)$. Clearly, $N(A)$ is a partially ordered set under fuzzy set inclusion.

Theorem 4.5. *Let* μ *be a fuzzy obstinate ideal of* A, $\mu(1) \neq 0$ *and* $\bar{\mu}$ *be the fuzzy set of* A, *defined by* $\tilde{\mu}(x) = \frac{\mu(x)}{\mu(1)}$, *for all* $x \in A$. *Then,* $\tilde{\mu}$ *is a normal fuzzy obstinate ideal of* A *and* $\mu \subseteq \tilde{\mu}$.

Proof. Let $x, y \in A$. We have

$$\tilde{\mu}(0) = \frac{\mu(0)}{\mu(1)} \geq \frac{\mu(x)}{\mu(1)} = \tilde{\mu}(x).$$

Therefore, $\tilde{\mu}(0) \geq \tilde{\mu}(x)$. Also, we have

$$\tilde{\mu}(x^*) = \frac{\mu(x^*)}{\mu(1)} \geq \frac{1 - \mu(x)}{\mu(1)} = \frac{1}{\mu(1)} - \frac{\mu(x)}{\mu(1)} \geq 1 - \frac{\mu(x)}{\mu(1)} = 1 - \tilde{\mu}(x)$$

Hence, $\tilde{\mu}$ is a fuzzy obstinate ideal of A. Clearly, $\tilde{\mu}$ is normal and $\mu \subseteq \tilde{\mu}$. □

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FUZZY OBSTINATE IDEALS IN MV-ALGEBRAS

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ایده‌آل‌های سرسخت فازی در MV-جبرها

فرشته فروزش
مجتمع آموزش عالی بم

در این مقاله، مفهوم ایده‌آل‌های سرسخت فازی در MV-جبرها را معرفی کرده و بعضی از ویژگی‌های آن‌ها را بیان می‌کنیم. ساختارهایی برای ایده‌آل‌های سرسخت فازی ارائه می‌دهیم. همچنین قضیه توسیع برای ایده‌آل‌های سرسخت فازی در یک MV-جبر A را ثابت می‌کنیم. رابطه بین ایده‌آل‌های سرسخت فازی و دیگر ایده‌آل‌های فازی در یک MV-جبر را مورد بررسی قرار می‌دهیم. اصل انتخاب برای ایده‌آل‌های سرسخت فازی در زیرمجموعه‌های تراز را توصیف می‌کنیم. بعلاوه، نشان می‌دهیم اگر μ یک ایده‌آل فازی سرسخت از A باشد به قسمی که $\mu(۰) \in [۰, ۱/۲]$، آنگاه A/μ یک جبر بولی است. در نهایت، مفهوم یک ایده‌آل سرسخت فازی نرمال را تعریف کرده و بعضی از ویژگی‌های آن را مورد بررسی قرار می‌دهیم.

کلمات کلیدی: MV-جبر، نرمال فازی، سرسخت فازی، بولی فازی.