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RADICAL OF FILTERS IN RESIDUATED LATTICES

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Abstract. In this paper, the notion of the radical of a filter in residuated lattices is defined and several characterizations of the radical of a filter are given. We show that if F is a positive implicative filter (or obstinate filter), then $Rad(F) = F$. We proved the extension theorem for radical of filters in residuated lattices. Also, we study the radical of filters in linearly ordered residuated lattices.

1. Introduction

Non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices. The integral commutative residuated l-monoid (i.e., residuated lattice), is an important class of logical algebras. Residuated lattices, introduced by Ward and Dilworth in [10], are a common structure among algebras associated with logical systems. The filter theory of the logical algebras plays an important role in studying these algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various filters correspond to various sets of provable formulas.

Busneag and Piciu in [5] introduced (positive) implicative and fantastic filters of residuated lattices. Ahadpanah and Torkzadeh, defined the notion of normal filters of residuated lattices in [1], and Bourmand

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Saeid and Pourkhatoun defined the notion of obstinate filters of residuated lattices in [3]. The aim of this paper is to present some new results in the field of residuated lattices, specifically by introducing and studying the radical of filters in residuated lattices.

The structure of this paper is as follows: In Section 2, we recall some definitions and facts about residuated lattices that we will use in the sequel. In Section 3, we will introduce the concept of the radical of a filter and we investigate some of its properties.

2. **Preliminaries**

A *residuated lattice* is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

(LR_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(LR_2) $(L, \odot, 1)$ is a commutative ordered monoid;
(LR_3) $\odot$ and $\rightarrow$ form an adjoint pair i.e, $c \leq a \rightarrow b$ if and only if $a \odot c \leq b$, for all $a, b, c \in L$.

Letting $x \in L$ be an arbitrary element, x^* is defined by $x \rightarrow 0$.

Proposition 2.1. [2, 7, 10] *Let L be a residuated lattice. Then, for any $x, y, z, w \in L$, we have:*

(R_1) $1 \rightarrow x = x$, $x \rightarrow x = 1$;
(R_2) $x \odot y \leq x, y$ *hence* $x \odot y \leq x \wedge y$, $x \leq y \rightarrow x$ *and* $x \odot 0 = 0$;
(R_3) $x \leq y$ *if and only if* $x \rightarrow y = 1$;
(R_4) $x \rightarrow 1 = 1$, $0 \rightarrow x = 1$, $1 \rightarrow 0 = 0$;
(R_5) $x \leq (x \rightarrow y) \rightarrow y$;
(R_6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y) \leq z \rightarrow (x \rightarrow y)$;
(R_7) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ *and* $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$;
(R_8) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$, $x \odot z \leq y \odot z$, $y^* \leq x^*$, *and* $x^{**} \leq y^{**}$;
(R_9) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$ *(so,* $x \rightarrow y^* = y \rightarrow x^* = (x \odot y)^*$*)*;
(R_{10}) $x \leq x^{**}$, $x^{***} = x^*$ *and* $x \leq x^* \rightarrow y$;
(R_{11}) $x \odot x^* = 0$, $x \odot y = 0$ *iff* $x \leq y^*$;
(R_{12}) $x^* \odot y^* \leq (x \odot y)^*$ *so,* $(x^*)^n \leq (x^n)^*$, *for every* $n \geq 1$;
(R_{13}) $x^{**} \odot y^{**} \leq (x \odot y)^{**}$ *so,* $(x^{**})^n \leq (x^n)^{**}$, *for every* $n \geq 1$;
(R_{14}) $(x \vee y)^* = x^* \wedge y^*$;
(R_{15}) $(x \rightarrow y^{**})^{**} = x \rightarrow y^{**}$;
(R_{16}) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

From now onwards, $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ or simply L, is a residuated lattice.

The following definitions are stated from [1, 3, 5, 9]. Let $\phi \neq F \subseteq L$, and $x, y, z \in L$. For convenience, we enumerate some conditions which will be used in the sequel:

(F_1) $x, y \in F$ implies $x \odot y \in F$ and $x \in F$, $x \leq y$ imply $y \in F$.
$(F_1)'$ $1 \in F$ and $x, x \to y \in F$ imply $y \in F$.
(F_2) $x \vee y \in F$ implies $x \in F$ or $y \in F$.
$(F_2)'$ $x \to y \in F$ or $y \to x \in F$.
(F_3) $x \notin F$ if and only if there exists $n \geq 1$ such that $(x^n)^* \in F$.
(F_4) $(y \to z) \to y \in F$ implies $y \in F$.
(F_5) $x, y \notin F$ implies $x \to y \in F$ and $y \to x \in F$.

F is called a *filter* of L, if it satisfies in the condition (F_1). The set of all filters in L, is denoted by $F(L)$. We have $F \in F(L)$ if and only if it satisfies in the condition $(F_1)'$. $F \in F(L)$ is called *proper* if $F \neq L$ (that is, $0 \notin F$). F is called a *prime filter* of L, if $0 \notin F$ and it satisfies in the conditions (F_1) and (F_2). We denote by $Spec(L)$, the set of all prime filters of L. $F \in Spec(L)$ if and only if $0 \notin F$ and it satisfies in conditions (F_1) and $(F_2)'$. F is called a *maximal filter* of L, if $0 \notin F$ and it satisfies in the conditions (F_1) and (F_3). We denote by $Max(L)$, the set of all maximal filters of L. F is called a *positive implicative filter* of L, if it satisfies in the conditions (F_1) and (F_4). We denote by $PIF(L)$, the set of all positive implicative filters of L. F is called an *obstinate filter* of L, if $0 \notin F$ and it satisfies in the conditions (F_1) and (F_5). We denote by $OF(L)$, the set of all obstinate filters of L. We have, $Max(L) \subseteq Spec(L)$.

Theorem 2.2. [8] *Let L be a nontrivial residuated lattice and $F \in F(L)$. Then*

(1) There exists $M \in Max(L)$, such that $F \subseteq M$.

(2) If $a \notin F$, there exists $P \in Spec(L)$ such that $F \subseteq P$ and $a \notin P$.

An element $a \in L$ is called *complemented* if there exists an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$. We will denote the set of all complemented elements in L by $B(L)$. If $e \in B(L)$, then $(e \to x) \to e = e$, for every $x \in L$, [4].

Definition 2.3. [6] The intersection of all maximal filters of a residuated lattice L is called the radical of L, and is denoted by $Rad(L)$. Then, $Rad(L) = \{a \in L : \ (a^n)^* \leq a, \text{ for any } n \in N\}$.

3. On Radical of Filters

Definition 3.1. Let F be a proper filter of L. The intersection of all maximal filters of L which contain F is called the radical of F, and it is denoted by $Rad(F)$. If $F = L$, then we put $Rad(L) = L$.

Note. $Rad(\{1\})$ is the same as $Rad(L)$, which is defined in [6].

Theorem 3.2. *Let $F \in F(L)$. Then*

(1) *$Rad(F) \in F(L)$.*
(2) *$F \subseteq Rad(F)$.*
(3) *If $F \in Max(L)$, then $Rad(F) = F$.*

Proof. By Definition 3.1, the proof is clear. □

In the following example, we show that the inverse inclusion of Theorem 3.2(3), may not hold in general.

Example 3.3. Let $L = \{0, a, b, c, d, 1\}$, where $0 < c < a, b < 1$ and $0 < d < a < 1$. Define $\odot$ and $\to$ as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	b	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

Then, $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice. We can see that $Rad(\{1\}) = \{1\}$, while $\{1\} \notin Max(L)$.

Theorem 3.4. *Let F be a proper filter of L and $a \in L$. The following conditions are equivalent:*

(1) *$a \in Rad(F)$;*
(2) *$(a^n)^* \to a \in F$, for all $n \in N$;*
(3) *$a^* \to a^n \in F$, for all $n \in N$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in Rad(F)$ and there exists $n \in N$ such that $(a^n)^* \to a \notin F$. Then, by Theorem 2.2(2), there exists $P \in Spec(L)$ such that $F \subseteq P$ and $(a^n)^* \to a \notin P$. Since $P \in Spec(L)$, we obtain $a \to (a^n)^* \in P$. Also, by Theorem 2.2(1), there exists $M \in Max(L)$ such that $M \supseteq P$. Therefore, $a \to (a^n)^* \in M$. If $a \in M$, then $a^n \in M$, for all $n \in N$. Also, we have $a \to (a^n)^* \in M$, hence $(a^n)^* \in M$ and so $0 = a^n \odot (a^n)^* \in M$, which is a contradiction. Thus, $a \notin M$. We have $F \subseteq P \subseteq M$ and $a \notin M$, hence $a \notin Rad(F)$, which is a contradiction. Therefore, $(a^n)^* \to a \in F$, for all $n \in N$.

(2) $\Rightarrow$ (1) Let $(a^n)^* \to a \in F$, for all $n \in N$ and $a \notin Rad(F)$. Then, there exists $M \in Max(L)$, such that $M \supseteq F$ and $a \notin M$. Since $M \in Max(L)$, there exists $n \in N$ such that $(a^n)^* \in M$. We have $(a^n)^* \to a \in F \subseteq M$, hence $a \in M$. Then, $a^n \in M$ and so $0 = (a^n)^* \odot a^n \in M$, which is a contradiction. Therefore, $a \in Rad(F)$.

(1) $\Rightarrow$ (3) Let $a \in Rad(F)$. Since $Rad(F) \in F(L)$, we obtain $a^n \in Rad(F)$, for all $n \in N$. So, by (1) $\Leftrightarrow$ (2), we get $((a^n)^m)^* \to a^n \in F$, for all $m \in N$. We have $a^{nm} \leq a$ then by Proposition 2.1, $(a^{nm})^* \to a^n \leq a^* \to a^n$. Therefore, $a^* \to a^n \in F$, for all $n \in N$.

(3) $\Rightarrow$ (1) Let $a^* \to a^n \in F$, for all $n \in N$ and $a \notin Rad(F)$. Then there exists $M \in Max(L)$ such that $F \subseteq M$ and $a \notin M$. Hence, there exists $m \in N$ such that $(a^m)^* \in M$. By Proposition 2.1, we have $a^m \leq (a^m)^{**}$, hence $a^* \to a^m \leq a^* \to (a^m)^{**}$, and so $a^* \to (a^m)^{**} \in F$. By Proposition 2.1, we have

$$\begin{aligned}(a^m)^* \to a^{**} &= (a^m)^* \to (a^* \to 0),\\ &= a^* \to ((a^m)^* \to 0),\\ &= a^* \to (a^m)^{**} \in F.\end{aligned}$$

Therefore, $(a^m)^* \to a^{**} \in F \subseteq M$. Since $(a^m)^* \in M$, we get that $a^{**} \in M$ and so $(a^{**})^m \in M$, for all $m \in N$. By Proposition 2.1, we have $(a^{**})^m \leq (a^m)^{**}$. Thus $(a^m)^{**} \in M$. Since $(a^m)^* \in M$, hence $0 = (a^m)^{**} \odot (a^m)^* \in M$, which is a contradiction, and our proof is finished. □

Theorem 3.5. *Let $F \in F(L)$. Then*

(1) *If $F \in PIF(L)$, then $Rad(F) = F$.*
(2) *If $F \in OF(L)$, then $Rad(F) = F$.*

Proof. (1) Let $F \in PIF(L)$. By Theorem 3.2(2), we must show that $Rad(F) \subseteq F$. Let $x \in Rad(F)$. Then, by Theorem 3.4, we get $(x^n)^* \to x \in F$, for all $n \in N$. Take $n = 1$, $(x \to 0) \to x \in F$. Thus, by the fact that $F \in PIF(L)$, we get $x \in F$, that is $Rad(F) \subseteq F$. Therefore, $Rad(F) = F$.

(2) The proof follows from $OF(L) \subseteq PIF(L)$ [3, Theorem 3.13] and part (1). □

Remark. By Theorem 3.4[1], if $F \in PIF(L)$, then $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$, for $x, y \in L$.

By the above remark and the following example, we conclude that the converse of Theorem 3.5, may not hold in general.

Example 3.6. Let $L = [0, 1]$. Define $\odot$ and $\to$, as follows:

$$x \odot y = min\{x, y\} \text{ and } x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases},$$

Then $(\mathrm{L}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice, and $F = [\frac{1}{2}, 1] \in F(L)$. We have $(\frac{1}{4} \rightarrow \frac{1}{5}) \rightarrow \frac{1}{5} = 1 \in F$ but $(\frac{1}{5} \rightarrow \frac{1}{4}) \rightarrow \frac{1}{4} = \frac{1}{4} \notin F$. Hence, $F \notin PIF(L)$ and so (by $OF(L) \subseteq PIF(L)$ [3, Theorem 3.13]) $F \notin OF(L)$, while $Rad(F) = F$.

Theorem 3.7. *Let F be a proper filter of L and $a, b \in Rad(F)$. Then, the following conditions hold:*

(1) $a^* \rightarrow b \in F$.
(2) $(a^* \odot b^*)^* \in F$.

Proof. (1) Let $a, b \in Rad(F)$. Then, $a \odot b \in Rad(F)$ and so $(a \odot b)^* \rightarrow (a \odot b) \in F$. We have $a \odot b \leq a$ then $(a \odot b)^* \rightarrow (a \odot b) \leq a^* \rightarrow (a \odot b)$. Therefore, $a^* \rightarrow (a \odot b) \in F$. Since $a \odot b \leq b$, then $a \odot b \rightarrow b = 1 \in F$ and so $(a^* \rightarrow (a \odot b)) \odot ((a \odot b) \rightarrow b) \in F$. By Proposition 2.1, $(a^* \rightarrow (a \odot b)) \odot ((a \odot b) \rightarrow b) \leq a^* \rightarrow b$. Hence, we obtain $a^* \rightarrow b \in F$.

(2) Let $a, b \in Rad(F)$. Then by (1), we have $a^* \rightarrow b \in F$. By Proposition 2.1, $b \leq b^{**}$ we get that $a^* \rightarrow b \leq a^* \rightarrow b^{**}$ and then $a^* \rightarrow b^{**} \in F$. By Proposition 2.1, we have $(a^* \odot b^*)^* = a^* \rightarrow b^{**}$ and then $(a^* \odot b^*)^* \in F$. □

Example 3.8. Let $L = \{0, a, b, c, d, 1\}$ be as in Example 3.3. We have $F = \{b, 1\} \in F(L)$. $d^* \rightarrow b = 1 \in F$ while $d \notin Rad(F) = \{b, 1\}$, hence the converse of Theorem 3.7(1) is not true in general. $(a^* \odot b^*)^* = 1 \in F$, while $a \notin Rad(F)$, hence the converse of Theorem 3.7(2) is not true in general.

Lemma 3.9. *Let F be a proper filter of L and $a \in L$. Then $a^* = 0$, for all $a \in L \setminus \{0\}$, if and only if $Rad(F) = L \setminus \{0\}$.*

Proof. Let $a^* = 0$, for all $a \in L \setminus \{0\}$. It is clear that $Rad(F) \subseteq L \setminus \{0\}$. We must show that $L \setminus \{0\} \subseteq Rad(F)$. Take $x \in L \setminus \{0\}$, then by hypothesis $x^* = 0$ and so $x^* \rightarrow x^n = 0 \rightarrow x^n = 1 \in F$, for all $n \in N$. Therefore, $x \in Rad(F)$, by Theorem 3.4. Hence, $Rad(F) = L \setminus \{0\}$.

Conversely, let $Rad(F) = L \setminus \{0\}$ and there exists $a \in L \setminus \{0\}$ such that $a^* \neq 0$. Hence, by hypothesis $a^*, a \in Rad(F)$, so $0 \in Rad(F)$, which is a contradiction. □

Theorem 3.10. *Let F and G be proper filters of L and $a, b \in L$. Then,*

(1) *If $F \subseteq G$, then $Rad(F) \subseteq Rad(G)$.*
(2) *$Rad(F) = L$ if and only if $F = L$.*

Proof. (1) Let $F \subseteq G$ and $x \in Rad(F)$. Then, $(x^n)^* \rightarrow x \in F \subseteq G$, for all $n \in N$. Hence, $x \in Rad(G)$.

(2) Let $Rad(F) = L$. Then, $0 \in Rad(F)$ and so $0 = 1 \rightarrow 0 = (0^n)^* \rightarrow 0 \in F$, for all $n \in N$. Therefore, $F = L$. The converse is clear. □

An element $a \in L$ is called a nilpotent element of L, if $a^n = 0$, for some $n \in N$. The set of all nilpotent elements of L is denoted by $Nil(L)$.

The order of $x \in L$, denoted by $ord(x)$, is the smallest $n \in N$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$.

Theorem 3.11. *Let L be a linear ordered residuated lattice and F be a proper filter of L. Then, we have the following statements.*

(1) *If $a \in Rad(F)$, then, $((a^n)^* \odot (a^n)^*) \to a = 1$, for all $n \in N$;*
(2) *If $a \notin Rad(F)$ then $a \in Nil(L)$;*
(3) *$Rad(F) = \{a : ord(a) = \infty\}$;*
(4) *$Rad(F) = \{a \in L : \ ((a^n)^*)^m = 0, \ \forall\, n \in N, \ \exists m \in N\}$.*

Proof. (1) Let $a \in Rad(F)$. Then $(a^n)^* \to a \in F$, for all $n \in N$. We have $(a^n)^* \to a \leq (a^n)^*$ or $(a^n)^* \leq (a^n)^* \to a$, for all $n \in N$. Let $(a^n)^* \to a \leq (a^n)^*$. Since $(a^n)^* \to a \in F$ then $(a^n)^* \in F$ and so $a \in F$. Hence, $a^n \in F$, for all $n \in N$, so $(a^n)^* \odot a^n \in F$. Therefore, $0 \in F$, which is a contradiction. Hence, $(a^n)^* \leq (a^n)^* \to a$, for all $n \in N$. Then $(a^n)^* \to ((a^n)^* \to a) = 1$, for all $n \in N$, so $((a^n)^* \odot (a^n)^*) \to a = 1$, for all $n \in N$.

(2) Let $a \notin Rad(F)$. Then, by Theorem 3.4, there exists $m \in N$, such that $(a^m)^* \to a \notin F$. Hence, $a < (a^m)^*$, and so by (LR_3), $a^{m+1} = a \odot a^m = 0$. Therefore, $a \in Nil(L)$.

(3) Let $a \in Rad(F)$ and $ord(a) < \infty$. Hence, there exists $m \in N$ such that $a^m = 0$. By filter property of $Rad(F)$, we get that $a^m \in Rad(F)$. Therefore, $0 \in Rad(F)$, which is a contradiction. Hence, $ord(a) = \infty$.

Conversely, let $ord(a) = \infty$ and $a \notin Rad(F)$. Then by (2), $a \in Nil(L)$, i.e. $ord(a) < \infty$. It is a contradiction, hence $a \in Rad(F)$. Thus, the proof is complete.

(4) Let $((a^n)^*)^m = 0$, for all $n \in N$, for some $m \in N$ and $a \notin Rad(F)$. Then, there exists $M \in Max(L)$ such that $F \subseteq M$ and $a \notin M$. So $(a^n)^* \in M$, for some $n \in N$. By hypothesis, we have $((a^n)^*)^m = 0$, for some $m \in N$, hence $0 \in M$, which is a contradiction. Therefore, $a \in Rad(F)$.

Conversely, let $a \in Rad(F)$, $((a^n)^*)^m \neq 0$, for some $n \in N$ and for all $m \in N$. Hence $ord((a^n)^*) = \infty$. By part (3), we obtain $(a^n)^* \in Rad(F)$, and we have $a^n \in Rad(F)$, for all $n \in N$. Therefore $0 = (a^n)^* \odot a^n \in Rad(F)$, which is a contradiction. So $((a^n)^*)^m = 0$, for all $n \in N$ and for some $m \in N$. □

Lemma 3.12. *Let $F \in F(L)$. Then*

(1) *$Rad(F) \bigcap B(L) \subseteq F$.*

(2) $Rad(\{1\}) \bigcap B(L) = \{1\}$.

Proof. (1) Let $x \in Rad(F) \bigcap B(L)$. Then, $x \in Rad(F)$ and $x \in B(L)$. By $x \in B(L)$, we have $(x \to 0) \to x = x$. By $x \in Rad(F)$, $(x^n)^* \to x \in F$, for all $n \in N$, and so $x^* \to x \in F$. We have $x^* \to x = x$. Hence, $x \in F$. Therefore, $Rad(F) \bigcap B(L) \subseteq F$.

(2) taking $F = \{1\}$ in part (1), the proof is clear. Then the proof is clear. □

In the following example we show that the equality of Lemma 3.12(1) may not hold, in general.

Example 3.13. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then, $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice and it is clear that $F = \{b, 1\} \in F(L)$ and $B(L) = \{0, 1\}$. Hence, $F \neq Rad(F) \bigcap B(L)$.

If $F \in F(L)$, then the relation $\sim_F$ defined on L by $(x, y) \in \sim_F$ if and only if $x \to y \in F$ and $y \to x \in F$ is a congruence relation on L. The *quotient algebra* $L/\sim_F$ denoted by L/F becomes a residuated lattice in a natural way, with the operations induced from those of L. So, the order relation on L/F is given by $x/F \leq y/F$ if and only if $x \to y \in F$. We have $G/F \in Max(L/F)$ if and only if $G \in Max(L)$ and $F \subseteq G$.

Theorem 3.14. *Let $F \in F(L)$. Then, we have the following statements.*

(1) $Rad(Rad(F)) = Rad(F)$.
(2) $Rad(Rad(F)/F) = Rad(F)/F = Rad(\{1\}/F)$.
(3) *If* $Rad(F) \subseteq B(L)$, *then* $Rad(F) = F$.

Proof. (1) By Theorem 3.2(2), we have $Rad(F) \subseteq Rad(Rad(F))$. It is enough to show that $Rad(Rad(F)) \subseteq Rad(F)$. Let $x \in Rad(Rad(F))$. Then, $x \in M$, for all $M \in Max(L)$ containing $Rad(F)$. Let $M_0 \in Max(L)$ containing F. Then $M_0 = Rad(M_0) \supseteq Rad(F)$ and so $x \in M_0$. Therefore, $x \in Rad(F)$, that is $Rad(Rad(F)) \subseteq Rad(F)$. Thus, $Rad(Rad(F)) = Rad(F)$.

(2) We have $Rad(F)/F \subseteq Rad(Rad(F)/F)$. We show that $Rad(Rad(F)/F) \subseteq Rad(F)/F$. Take $a/F \in Rad(Rad(F)/F)$, then $((a/F)^n)^* \to a/F \in Rad(F)/F$, for all $n \in N$. Hence, $((a^n)^* \to a)/F = b/F$, for some $b \in Rad(F)$, so $b \to ((a^n)^* \to a) \in F \subseteq Rad(F)$. Therefore, $(a^n)^* \to a \in Rad(F)$, for all $n \in N$, that is $a \in Rad(Rad(F))$. Thus, $a/F \in Rad(Rad(F))/F = Rad(F)/F$.

Now, by definition of radical, we have

$$Rad(\{1\}/F) = \bigcap_{\substack{N \in Max(L) \\ F \subseteq N}} (N/F) = (\bigcap_{\substack{N \in Max(L) \\ F \subseteq N}} N)/F = Rad(F)/F.$$

(3) It is clear by Lemma 3.12(1). □

Proposition 3.15. *Let $F \in F(L)$. Then $Rad(F) = F$ if and only if $Rad(\{1\}/F) = \{1\}/F$.*

Proof. Let $Rad(F) = F$. Then, by Theorem 3.14(2), we have $Rad(\{1\}/F) = Rad(F)/F = F/F = \{1\}/F$. Hence $Rad(\{1\}/F) = \{1\}/F$.

Conversely, let $Rad(\{1\}/F) = \{1\}/F$. Then by Theorem 3.14(2), $Rad(F)/F = \{1\}/F$. We must show that $Rad(F) \subseteq F$. Let $x \in Rad(F)$. Then, $x/F \in Rad(F)/F = \{1\}/F$, and $x/F = 1/F$ that is $x \in F$, hence $Rad(F) \subseteq F$. Therefore, $Rad(F) = F$. □

Proposition 3.16. *Let L be a linear residuated lattice. $a \in Nil(L)$ if and only if $a/Rad(F) \in Nil(L/Rad(F))$.*

Proof. Let L be a linear residuated lattice and $a/Rad(F) \in Nil(L/Rad(F))$. Then there exists $n \in N$ such that $a^n/Rad(F) = (a/Rad(F))^n = 0/Rad(F)$, and so $a^n \notin Rad(F)$. By Theorem 3.4, there exists $m \in N$ such that $((a^n)^m)^* \to a^n \notin F$, hence $((a^n)^m)^* \nleq a^n$. By hypothesis, we get that $a^n < ((a^n)^m)^*$, so by (LR_3), we get that $a^n \odot (a^n)^m = 0$. Therefore, $a \in Nil(L)$.

Conversely, let $a \in Nil(L)$. Then, there exists $n \in N$ such that $a^n = 0$. Hence $0/Rad(F) = a^n/Rad(F) = (a/Rad(F))^n$. Therefore, $a/Rad(F) \in Nil(L/Rad(F))$. □

Proposition 3.17. *Let $\{F_i\}_{i \in I}$ be a family of filters of L. Then, $Rad(\bigcap_{i \in I} F_i) = \bigcap_{i \in I} Rad(F_i)$.*

Proof. We have $\bigcap_{i \in I} F_i \subseteq F_i \subseteq Rad(F_i)$, for all $i \in I$, then by Theorems 3.10(1) and 3.14(1), we get that $Rad(\bigcap_{i \in I} F_i) \subseteq Rad(F_i)$, for all $i \in I$. Therefore, $Rad(\bigcap_{i \in I} F_i) \subseteq \bigcap_{i \in I}(Rad(F_i))$.

Conversely, let $x \in \bigcap_{i \in I}(Rad(F_i))$. Then $x \in Rad(F_i)$, for all $i \in I$, and so $(x^n)^* \to x \in F_i$, for all $i \in I$ and $n \in N$. Hence, $(x^n)^* \to x \in \bigcap_{i \in I} F_i$, for all $n \in N$, that is $x \in Rad(\bigcap_{i \in I} F_i)$. Therefore, $Rad(\bigcap_{i \in I} F_i) = \bigcap_{i \in I} Rad(F_i)$. □

Theorem 3.18. *Let $x \wedge x^* = 0$, for all $x \in L$. We have the following statements:*

(1) $Nil(L) = \{0\}$.
(2) *If L is a linear residuated lattice, then $Rad(F) = \{x \in L : x^* = 0\}$, for each proper filter F of L.*

Proof. (1) Suppose that there exists $0 \neq x \in Nil(L)$. So there is the smallest natural number n such that $x^n = 0$. Hence, by Proposition 2.1, we get that $x^{n-1} \leq x^*$. On the other hand, we have $x^{n-1} \leq x$, so $x^{n-1} \leq x^* \wedge x = 0$. Hence, $x^{n-1} = 0$, which is a contradiction. Therefore, 0 is the only nilpotent element of L, i.e. $Nil(L) = \{0\}$.

(2) Let $x \in Rad(F)$. Then, by Theorem 3.11(4), we have $((x^n)^*)^m = 0$, for all $n \in N$ and for some $m \in N$. So, $(x^n)^* \in Nil(L)$, for all $n \in N$. Thus by part (1), we get that $x^* = 0$.

Conversely, let $x^* = 0$. Then, $0 = x^* \leq x^n$, for all $n \in N$, and so $x^* \to x^n = 1 \in F$, for all $n \in N$. Hence, $x \in Rad(F)$, by Theorem 3.4. □

4. Conclusion

In this paper, we introduced the notion of the radical of a filter F in residuated lattices and we presented a characterization and many important properties of $Rad(F)$. We proved that if $F \in PIF(L)$ or $F \in OF(L)$, then $Rad(F) = F$. Finally, we proved that in linearly ordered residuated lattice, radical of all proper filters are equal.

In our future work, we are going to consider the notion of the radical of primary filters and try to define other types of filters in residuated lattices and other logical algebraic structures. We hope this work would serve as a foundation for further studies on the structure of residuated lattices and develop corresponding many-valued logical systems.

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RADICAL OF FILTERS IN RESIDUATED LATTICES

S. MOTAMED

رادیکال فیلترها در مشبکه‌های مانده

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در این مقاله، مفهوم رادیکال یک فیلتر در مشبکه‌های مانده تعریف شده است و ویژگی‌های آن بدست آمده است. نشان داده‌ایم اگر فیلتر استلزامی مثبت (یا سرسخت) باشد، آنگاه رادیکال فیلتر با فیلتر برابر است و ویژگی توسیع را برای رادیکال فیلترها در مشبکه‌های مانده ثابت کردیم. همچنین ویژگی رادیکال فیلترها را در مشبکه‌های مانده خطی بررسی کردیم.

کلمات کلیدی: فیلتر اول (ماکسیمال)، رادیکال، مشبکه‌های مانده.